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Stochastic Navier–Stokes Equations Driven by Lévy Noise in Unbounded 3D Domains

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Abstract Martingale solutions of the stochastic Navier–Stokes equations in 2D and 3D possibly unbounded domains, driven by the Lévy noise consisting of the compensated time homogeneous Poisson random measure and the Wiener process are considered. Using the classical Faedo–Galerkin approximation and the compactness method we prove existence of a martingale solution. We prove also the compactness and tightness criteria in a certain space contained in some spaces of *càdlàg* functions, *weakly càdlàg* functions and some Fréchet spaces. Moreover, we use a version of the Skorokhod Embedding Theorem for nonmetric spaces.

Keywords Stochastic Navier–Stokes equations · Martingale solution · Poisson random measure · Compactness method

Mathematics Subject Classifications (2010) 35Q30 · 60H15 · 76M35

1 Introduction

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open connected possibly unbounded subset with smooth boundary $\partial\mathcal{O}$, where $d = 2, 3$. We will consider the Navier–Stokes equations

$$du(t) = \big[\Delta u - (u \cdot \nabla)u + \nabla p + f(t)\big]\, dt + \int_Y F(t,u)\, \tilde{\eta}(dt, dy)$$
$$+ G(t, u(t))\, dW(t), \qquad t \in [0,T], \tag{1}$$

in $\mathcal{O}$, with the incompressibility condition

$$\operatorname{div} u = 0, \tag{2}$$

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the initial condition

$$u(0) = u_0, \tag{3}$$

and with the homogeneous boundary condition $u_{|\partial\mathscr{O}} = 0$. In this problem $u = u(t,x) = (u_1(t,x), \dots u_d(t,x))$ and $p = p(t,x)$ represent the velocity and the pressure of the fluid, respectively. Furthermore, f stands for the deterministic external forces. The terms $\int_Y F(t,u)\,\tilde{\eta}(dt,dy)$, where $\tilde{\eta}$ is a compensated time homogeneous Poisson random measure on a certain measurable space $(Y,\mathscr{Y})$, and $G(t,u(t))\,dW(t)$, where W is a cylindrical Wiener process on some separable Hilbert space Y_W, stand for the random forces.

The problem (1–3) can be written as the following stochastic evolution equation

$$\begin{aligned} du(t) \quad &+\mathscr{A}u(t)\,dt + B\big(u(t)\big)\,dt = f(t)\,dt + \int_Y F(t,u(t^-);y)\tilde{\eta}(dt,dy) \\ &+G(t,u(t))\,dW(t) \qquad t \in [0,T], \\ u(0) \quad &= u_0. \end{aligned}$$

We will prove the existence of a martingale solution of the problem (1–3) understood as a system $(\Omega,\mathscr{F},\mathbb{P},\mathbb{F},\eta,W,u)$, where $(\Omega,\mathscr{F},\mathbb{P},\mathbb{F})$ is a filtered probability space, η is a time homogeneous Poisson random measure, W is a cylindrical Wiener process and $u = (u_t)_{t\in[0,T]}$ is a stochastic process with trajectories in the space $\mathbb{D}\big([0,T],H_w\big) \cap L^2(0,T;V)$ and satisfying appropriate integral equality, see Definition 3 in Section 4.2. Here, V and H denote the closures in $H^1(\mathscr{O},\mathbb{R}^d)$ and $L^2(\mathscr{O},\mathbb{R}^d)$, respectively of the space $\mathscr{V}$ of the divergence-free $\mathbb{R}^d$ valued vector fields of class $\mathscr{C}^\infty$ with compact supports contained in $\mathscr{O}$. The symbol $\mathbb{D}\big([0,T],H_w\big)$ stands for the space of H valued weakly càdlàg functions.

To construct this solution we use the classical Faedo–Galerkin method, i.e.,

$$\begin{aligned} du_n(t) &= -\big[P^n\mathscr{A}u_n(t) + B_n\big(u_n(t)\big) - P_n f(t)\big]\,dt \\ &\qquad + \int_Y P_nF(t,u_n(t^-),y)\tilde{\eta}(dt,dy) + P_nG(t,u(t))\,dW(t), \;\; t\in[0,T], \\ u_n(0) &= P_nu_0. \end{aligned}$$

The solutions u_n to the Galerkin scheme generate a sequence of laws $\{\mathscr{L}(u_n), n\in\mathbb{N}\}$ on appropriate functional spaces. To prove that this sequence of probability measures is weakly compact we need appropriate tightness criteria.

We concentrate first on the compactness and tightness criteria. If the domain $\mathscr{O}$ is unbounded, then the embedding $V \subset H$ is not compact. However using Lemma 2.5 in [16], see Appendix C, we can find a separable Hilbert space U such that $U \subset V$, the embedding being dense and compact.

We consider the intersection

$$\mathscr{Z}_q := L^q_w(0,T;V) \cap L^q(0,T;H_{loc}) \cap \mathbb{D}([0,T];U') \cap \mathbb{D}([0,T],H_w),$$

where $q\in(1,\infty)$. (The letter w indicates the weak topology.) By $\mathbb{D}([0,T];U')$ we denote the space of U'-valued *càdlàg* functions equipped with the Skorokhod topology and $L^q(0,T;H_{\mathrm{loc}})$ stands for the Fréchet space defined by Eq. 24, see Section 3.2.

Using the compactness criterion in the space of *càdlàg* functions, we prove that a set $\mathscr{K}$ is relatively compact in $\mathscr{L}_q$ if the following three conditions hold

(a) for all $u \in \mathscr{K}$ and all $t \in [0, T]$, $u(t) \in H$ and $\sup_{u\in\mathscr{K}} \sup_{s\in[0,T]} |u(s)|_H < \infty$,
(b) $\sup_{u\in\mathscr{K}} \int_0^T \|u(s)\|_V^q \, ds < \infty$, i.e. $\mathscr{K}$ is bounded in $L^q(0, T; V)$,
(c) $\lim_{\delta\to 0} \sup_{u\in\mathscr{K}} w_{[0,T],U'}(u; \delta) = 0$.

Here $w_{[0,T],U'}(u; \delta)$ stands for the modulus of the function $u : [0, T] \to U'$. The above result is a straightforward generalization of the compactness results of [9] and [25]. In the paper [25] the analogous result is proved in the case when the embedding $V \subset H$ is dense and compact (in the Banach space setting). In [9] the embedding $V \subset H$ is only dense and continuous. However, instead of the spaces of càdlàg functions, appropriate spaces of continuous functions are used. The present paper generalizes both [9] and [25] in the sense that the embedding $V \subset H$ is dense and continuous and appropriate spaces of càdlàg functions are considered, i.e. $\mathbb{D}([0, T]; U')$ and $\mathbb{D}([0, T], H_w)$. This approach were strongly inspired by the results due to Métivier and Viot, especially the choice of the spaces $\mathbb{D}([0, T]; U')$ and $\mathbb{D}([0, T], H_w)$, see [23] and [22]. It is also closely related to the result due to Mikulevicius and Rozovskii [24] and to the classical Dubinsky compactness criterion, [28]. However, both in [28] and [24], the spaces of continuous functions are used.

Using the above deterministic compactness criterion and the Aldous condition in the form given by Joffe and Métivier [19], see also [22], we obtain the corresponding tightness criterion for the laws on the space $\mathscr{Z}_q$, see Corollary 1.

We will prove that the set of probability measures induced by the Galerkin solutions is tight on the space $\mathscr{Z}$, where

$$\mathscr{Z} := L^2_w(0, T; V) \cap L^2(0, T; H_{\text{loc}}) \cap \mathbb{D}([0, T]; U') \cap \mathbb{D}([0, T]; H_w),$$

which is not metrizable. Further construction a martingale solutions is based on the Skorokhod Embedding Theorem in nonmetric spaces. In fact, we use the result proved in [25] and following easily from the Jakubowski's version of the Skorokhod Theorem [18] and the version of the Skorokhod Theorem due to Brzeźniak and Hausenblas [6], see Appendix B. This will allow us to construct a stochastic process $\bar{u}$ with trajectories in the space $\mathscr{Z}$, a time homogeneous Poisson random measure $\bar{\eta}$ and a cylindrical Wiener proces $\bar{W}$ defined on some filtered probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}})$ such that the system $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}}, \bar{\eta}, \bar{W}, \bar{u})$ is a martingale solution of the problem (1–3). In fact, $\bar{u}$ is a process with trajectiories in the space $\mathscr{Z}$. In particular, the trajectories of $\bar{u}$ are *weakly càdlàg* if $\bar{u}$ is considered as a H-valued process and *càdlàg* in the bigger space U'.

The Navier–Stokes equations driven by the compensated Poisson random measure in the 3D bounded domains were studied in Dong and Zhai [15]. The authors consider the martingale problem associated to the Navier–Stokes equations, i.e. a solution is defined to be a probability measure satisfying appropriate conditions, see Definition 3.1 in [15]. The 2D Navier–Stokes equations were considered in [13, 14] and [29]. In the present paper, using a different approach we generalize the existence resuls to the case of unbounded 2D and 3D domains. Moreover, we consider more general noise term.

Stochastic Navier–Stokes equations in unbounded 2D and 3D domains were usually considered with the Gaussian noise term, see e.g. [8, 11, 12] and [9]. Martingale solutions of the stochastic Navier–Stokes equations driven by white noise in the whole space $\mathbb{R}^d$, $(d \geq 2)$, are investigated in [24].

The present paper is organized as follows. In Section 2 we recall basic definitions and properties of the spaces and operators appearing in the Navier–Stokes equations. Section 3 is devoted to the compactness and tightness results. Some auxilliary results about the Aldous condition and tightness are contained in Appendix A. Precise statement of the Navier–Stokes problem driven by Lévy noise is contained in Section 4.2. The main Theorem about existence of a martingale solution of the problem (1–3) is proved in Section 5. Some versions the Skorokhod Embedding Theorems are recalled in Appendix B. In Appendix C we recall Lemma 2.5 in [16] together with the proof.

2 Functional Setting

2.1 Basic Definitions

Let $\mathscr{O} \subset \mathbb{R}^d$ be an open connected subset with smooth boundary $\partial\mathscr{O}$, $d = 2, 3$. Let

$$\mathscr{V} := \{u \in \mathscr{C}_c^\infty(\mathscr{O}, \mathbb{R}^d) : \ \operatorname{div} u = 0\},$$

$$H := \text{the closure of } \mathscr{V} \text{ in } L^2(\mathscr{O}, \mathbb{R}^d), \tag{4}$$

$$V := \text{the closure of } \mathscr{V} \text{ in } H^1(\mathscr{O}, \mathbb{R}^d). \tag{5}$$

In the space H we consider the scalar product and the norm inherited from $L^2(\mathscr{O}, \mathbb{R}^d)$ and denote them by $(\cdot|\cdot)_H$ and $|\cdot|_H$, respectively, i.e.

$$(u|v)_H := (u|v)_{L^2}, \qquad |u|_H := \|u\|_{L^2}, \qquad u, v \in H.$$

In the space V we consider the scalar product inherited from the Sobolev space $H^1(\mathscr{O}, \mathbb{R}^d)$, i.e.

$$(u|v)_V := (u|v)_{L^2} + ((u|v)), \tag{6}$$

where

$$((u|v)) := (\nabla u|\nabla v)_{L^2}, \qquad u, v \in V. \tag{7}$$

and the norm

$$\|u\|_V^2 := |u|_H^2 + \|u\|^2, \tag{8}$$

where $\|u\|^2 := \|\nabla u\|_{L^2}^2$.

2.2 The Form b

Let us consider the following three-linear form, see Temam [27],

$$b(u, w, v) = \int_{\mathscr{O}} \big(u \cdot \nabla w\big) v \, dx.$$

We will recall those fundamental properties of the form b that are valid both in bounded and unbounded domains. By the Sobolev embedding Theorem, see [1], and the Hőlder inequality, we obtain the following estimates

$$|b(u,w,v)| \le c\|u\|_V \|w\|_V \|v\|_V, \qquad u,w,v \in V \tag{9}$$

for some positive constant c. Thus the form b is continuous on V, see also [27]. Moreover, if we define a bilinear map B by $B(u,w) := b(u,w,\cdot)$, then by inequality (9) we infer that $B(u,w) \in V'$ for all $u, w \in V$ and that the following inequality holds

$$|B(u,w)|_{V'} \le c\|u\|_V \|w\|_V, \qquad u,w \in V. \tag{10}$$

Moreover, the mapping $B : V \times V \to V'$ is bilinear and continuous. Let us also recall the following properties of the form b, see Temam [27], Lemma II.1.3,

$$b(u,w,v) = -b(u,v,w), \quad u,w,v \in V.$$

In particular,

$$b(u,v,v) = 0 \qquad u,v \in V.$$

Hence

$$\langle B(u,w)|v\rangle = -\langle B(u,v)|w\rangle, \qquad u,w,v \in V$$

and

$$\langle B(u,v)|v\rangle = 0, \qquad u,v \in V. \tag{11}$$

Let us, for any $m > 0$ define the following standard scale of Hilbert spaces

$$V_m := \text{the closure of } \mathscr{V} \text{ in } H^m(\mathscr{O},\mathbb{R}^d).$$

If $m > \frac{d}{2} + 1$ then by the Sobolev embedding Theorem, see [1],

$$H^{m-1}(\mathscr{O},\mathbb{R}^d) \hookrightarrow \mathscr{C}_b(\mathscr{O},\mathbb{R}^d) \hookrightarrow L^\infty(\mathscr{O},\mathbb{R}^d),$$

where $\mathscr{C}_b(\mathscr{O},\mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued continuous and bounded functions defined on $\mathscr{O}$. If $u, w \in V$ and $v \in V_m$ with $m > \frac{d}{2} + 1$ then

$$\begin{aligned} |b(u,w,v)| = |b(u,v,w)| &= \left| \sum_{i=1}^n \int_{\mathscr{O}} u_i w \frac{\partial v}{\partial x_i}\, dx \right| \\ &\le \|u\|_{L^2}\|w\|_{L^2}\|\nabla v\|_{L^\infty} \le c\|u\|_{L^2}\|w\|_{L^2}\|v\|_{V_m} \end{aligned}$$

for some constant $c > 0$. Thus, b can be uniquely extented to the three-linear form (denoted by the same letter)

$$b : H \times H \times V_m \to \mathbb{R}$$

and $|b(u,w,v)| \le c\|u\|_{L^2}\|w\|_{L^2}\|v\|V_m$ for $u, w \in H$ and $v \in V_m$. At the same time the operator B can be uniquely extended to a bounded bilinear operator

$$B : H \times H \to V_m'.$$

In particular, it satisfies the following estimate

$$|B(u,w)|_{V_m'} \le c|u|_H |w|_H, \qquad u,w \in H. \tag{12}$$

See Vishik and Fursikov [28]. We will also use the following notation, $B(u) := B(u,u)$. Let us also recall the well known result that the map $B: V \to V'$ is locally Lipschitz continuous, i.e. for every $r > 0$ there exists a constant L_r such that

$$\left|B(u) - B(\tilde{u})\right|_{V'} \le L_r \|u - \tilde{u}\|_V, \qquad u, \tilde{u} \in V, \quad \|u\|_V, \|\tilde{u}\|_V \le r. \tag{13}$$

2.3 The Space U and Some Operators

We recall operators and their properties used in [9]. Here we also recall the definition of a Hilbert space U compactly embedded in appropriate space V_m. This is possible thanks to the result due to Holly and Wiciak, [16] which we recall with the proof in Appendix C, see Lemma 10. This space will be of crucial importance in further investigations.

Consider the natural embedding $j: V \hookrightarrow H$ and its adjoint $j^*: H \to V$. Since the range of j is dense in H, the map j^* is one-to-one. Let us put

$$\begin{aligned} D(A) &:= j^*(H) \subset V, \\ Au &:= \left(j^*\right)^{-1} u, \qquad u \in D(A) \end{aligned} \tag{14}$$

and

$$\mathscr{A} u := ((u|\cdot)), \qquad u \in V, \tag{15}$$

where $((\cdot|\cdot))$ is defined by Eq. 7. Let us notice that if $u \in V$, then $\mathscr{A} u \in V'$ and

$$|\mathscr{A} u|_{V'} \le \|u\|.$$

Indeed, this follows immediately from Eq. 8 and the following inequalities

$$|((u|v))| \le \|u\| \cdot \|v\| \le \|u\| (\|v\|^2 + |v|_H^2)^{\frac{1}{2}} = \|u\| \cdot \|v\|_V, \quad v \in V.$$

Lemma 1 (Lemma 2.2 in [9])

(a) *For any $u \in D(A)$ and $v \in V$:*

$$((A - I)u|v)_H = ((u|v)) = \langle \mathscr{A} u | v \rangle,$$

where I stands for the identity operator on H and $\langle | \rangle$ is the standard duality pairing. In particular,

$$|\mathscr{A} u|_{V'} \le |(A - I)u|_H.$$

(b) *$D(A)$ is dense in H.*

Proof To prove assertion (a), let $u \in D(A)$ and $v \in V$. Then

$$\begin{aligned} (Au|v)_H &= ((j^*)^{-1}u|v)_H = ((j^*)^{-1}u|jv)_H = (j^*(j^*)^{-1}u|v)_V = (u|v)_V \\ &= (u|v)_H + ((u|v)) = (Iu|v)_H + \langle \mathscr{A} u|v \rangle. \end{aligned}$$

Let us move to the proof of part (b). Since V is dense in H, it is sufficient to prove that $D(A)$ is dense in V. Let $w \in V$ be an arbitrary element orthogonal to $D(A)$ with respect to the scalar product in V. Then

$$(u|w)_V = 0 \qquad \text{for} \quad u \in D(A).$$

On the other hand, by (a) and Eq. 6, $(u|w)_V = (Au|w)_H$ for $u \in D(A)$. Hence $(Au|w)_H = 0$ for $u \in D(A)$. Since $A : D(A) \to H$ is onto, we infer that $w = 0$, which completes the proof. □

Let us assume that $m > 1$. It is clear that V_m is dense in V and the embedding $j_m : V_m \hookrightarrow V$ is continuous. Then by Lemma 10 in Appendix C, there exists a Hilbert space U such that $U \subset V_m$, U is dense in V_m and

$$\textit{the natural embedding } \iota_m : U \hookrightarrow V_m \textit{ is compact}\,. \tag{16}$$

Then we have

$$U \overset{\iota_m}{\hookrightarrow} V_m \overset{j_m}{\hookrightarrow} V \overset{j}{\hookrightarrow} H \cong H' \overset{j'}{\hookrightarrow} V' \overset{j_m'}{\hookrightarrow} V_m' \overset{\iota_m'}{\hookrightarrow} U'. \tag{17}$$

Since the embedding ι_m is compact, ι_m' is compact as well. Consider the composition

$$\iota := j \circ j_m \circ \iota_m : U \hookrightarrow H$$

and its adjoint

$$\iota^* := (j \circ j_m \circ \iota_m)^* = \iota_m^* \circ j_m^* \circ j^* : H \to U.$$

Note that ι is compact and since the range of ι is dense in H, $\iota^* : H \to U$ is one-to-one. Let us put

$$\begin{aligned} D(L) &:= \iota^*(H) \subset U, \\ Lu \ &:= \left(\iota^*\right)^{-1} u, \qquad u \in D(L). \end{aligned} \tag{18}$$

It is clear that $L : D(L) \to H$ is onto. Let us also notice that

$$(Lu|w)_H = (u|w)_U, \qquad u \in D(L), \quad w \in U. \tag{19}$$

By equality (19) and the densiness of U in H, we infer similarly as in the proof of assertion (b) in Lemma 1 that $D(L)$ is dense in H. Moreover, for $u \in D(L)$,

$$Lu = \left(\iota^*\right)^{-1} u = \left(\iota_m^* \circ j_m^* \circ j^*\right)^{-1} u = A \circ \left(j_m^*\right)^{-1} \circ \left(\iota_m^*\right)^{-1} u,$$

where A is defined by Eq. 14.

Since L is self-adjoint and L^{-1} is compact, there exists an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H composed of the eigenvectors of operator L. Let us fix $n \in \mathbb{N}$ and let P_n be the operator from U' to $span\{e_1, ..., e_n\}$ defined by

$$P_n u^* := \sum_{i=1}^{n} \langle u^* | e_i \rangle e_i, \qquad u^* \in U', \tag{20}$$

where $\langle \cdot | \cdot \rangle$ denotes the duality pairing between the space U and its dual U'. Note that the restriction of P_n to H, denoted still by P_n, is given by

$$P_n u = \sum_{i=1}^{n} (u|e_i)_H e_i, \qquad u \in H,$$

and thus it is the $(\cdot|\cdot)_H$-orthogonal projection onto $span\{e_1, ..., e_n\}$. Restrictions of P_n to other spaces considered in Eq. 17 will also be denoted by P_n. Moreover, it is easy to see that

$$(P_n u^* | v)_H = \langle u^* | P_n v\rangle, \qquad u^* \in U', \quad v \in U.$$

It is easy to prove that the system $\left\{\frac{e_i}{\|e_i\|U}\right\}_{n\in\mathbb{N}}$ is the $(\cdot|\cdot)_U$-orthonormal basis in the space U and that the restriction of P_n to U is the $(\cdot|\cdot)_U$-projection onto the subspace $span\{e_1, ..., e_n\}$. In particular, for every $u \in U$

(i) $\lim_{n\to\infty} \| P_n u - u\|_U = 0$,
(ii) $\lim_{n\to\infty} \| P_n u - u\|_{V_m} = 0$, where $m > 0$,
(iii) $\lim_{n\to\infty} \| P_n u - u\|_V = 0$.

See Lemma 2.4 in [9] for details.

We will use the basis $\{e_i\}_{i\in\mathbb{N}}$ and the operators P_n in the Faedo–Galerkin approximation.

3 Compactness Results

3.1 The Space of Càdlàg Functions

Let $(\mathbb{S}, \rho)$ be a separable and complete metric space. Let $\mathbb{D}([0, T]; \mathbb{S})$ the space of all $\mathbb{S}$-valued *càdlàg* functions defined on $[0, T]$, i.e. the functions which are right continuous and have left limits at every $t \in [0, T]$. The space $\mathbb{D}([0, T]; \mathbb{S})$ is endowed with the Skorokhod topology.

Remark 1 A sequence $(u_n) \subset \mathbb{D}([0, T]; \mathbb{S})$ converges to $u \in \mathbb{D}([0, T]; \mathbb{S})$ iff there exists a sequence (λ_n) of homeomorphisms of $[0, T]$ such that λ_n tends to the identity uniformly on $[0, T]$ and $u_n \circ \lambda_n$ tends to u uniformly on $[0, T]$.

This topology is metrizable by the following metric δ_T

$$\delta_T(u, v) := \inf_{\lambda\in\Lambda_T} \left[\sup_{t\in[0,T]} \rho\big(u(t), v\circ\lambda(t)\big) + \sup_{t\in[0,T]} |t-\lambda(t)| + \sup_{s\neq t} \left|\log \frac{\lambda(t)-\lambda(s)}{t-s}\right| \right],$$

where Λ_T is the set of increasing homeomorphisms of $[0, T]$. Moreover, $\big(\mathbb{D}([0, T]; \mathbb{S}), \delta_T\big)$ is a complete metric space, see [19].

Let us recall the notion of a *modulus* of the function. It plays analogous role in the space $\mathbb{D}([0, T]; \mathbb{S})$ as *the modulus of continuity* in the space of continuous functions $\mathbb{C}([0, T]; \mathbb{S})$.

Definition 1 (see [22]) Let $u \in \mathbb{D}([0, T]; \mathbb{S})$ and let $\delta > 0$ be given. A **modulus** of u is defined by

$$w_{[0,T],\mathbb{S}}(u, \delta) := \inf_{\Pi_\delta} \max_{t_i\in\bar{\omega}} \sup_{t_i\leq s<t<t_{i+1}\leq T} \rho\big(u(t), u(s)\big), \tag{21}$$

where Π_δ is the set of all increasing sequences $\bar{\omega} = \{0 = t_0 < t_1 < ... < t_n = T\}$ with the following property

$$t_{i+1} - t_i \geq \delta, \qquad i = 0, 1, ..., n-1.$$

If no confusion seems likely, we will denote the modulus by $w_{[0,T]}(u, \delta)$.

We have the following criterion for relative compactness of a subset of the space $\mathbb{D}([0,T];\mathbb{S})$, see [19, 22], Ch.II, and [4], Ch.3, analogous to the Arzelà–Ascoli Theorem for the space of continuous functions.

Theorem 1 *A set $A \subset \mathbb{D}([0,T];\mathbb{S})$ has compact closure iff it satisfies the following two conditions:*

(a) *there exists a dense subset $J \subset [0,T]$ such that for every $t \in J$ the set $\{u(t),\ u \in A\}$ has compact closure in $\mathbb{S}$.*
(b) $\lim_{\delta \to 0} \sup_{u \in A} w_{[0,T]}(u, \delta) = 0$.

3.2 Deterministic Compactness Criterion

Let us recall that V and H are Hilbert spaces defined by Eqs. 4–8. Since $\mathcal{O}$ is an arbitrary domain of $\mathbb{R}^d$, $(d = 2, 3)$, the embedding $V \hookrightarrow H$ is dense and continuous. We have defined a Hilbert space $U \subset V$ such that the embedding $U \hookrightarrow V$ is dense and compact, see Eq. 16. In particular, we have

$$U \hookrightarrow V \hookrightarrow H \cong H' \hookrightarrow U',$$

the embedding $H \hookrightarrow U'$ being compact as well. Let $\left(\mathcal{O}_R\right)_{R \in \mathbb{N}}$ be a sequence of open and bounded subsets of $\mathcal{O}$ with regular boundaries $\partial \mathcal{O}_R$ such that $\mathcal{O}_R \subset \mathcal{O}_{R+1}$ and $\bigcup_{R=1}^{\infty} \mathcal{O}_R = \mathcal{O}$. We will consider the following spaces of restrictions of functions defined on $\mathcal{O}$ to subsets $\mathcal{O}_R$, i.e.

$$H_{\mathcal{O}_R} := \{u_{|\mathcal{O}_R};\ \ u \in H\} \qquad V_{\mathcal{O}_R} := \{v_{|\mathcal{O}_R};\ \ v \in V\} \tag{22}$$

with appropriate scalar products and norms, i.e.

$$(u|v)_{H_{\mathcal{O}_R}} := \int_{\mathcal{O}_R} uv\, dx, \qquad u, v \in H_{\mathcal{O}_R},$$
$$(u|v)_{V_{\mathcal{O}_R}} := \int_{\mathcal{O}_R} uv\, dx + \int_{\mathcal{O}_R} \nabla u \nabla v\, dx, \qquad u, v \in V_{\mathcal{O}_R}$$

and $|u|^2_{H_{\mathcal{O}_R}} := (u|u)_{H_{\mathcal{O}_R}}$ for $u \in H_{\mathcal{O}_R}$ and $\|u\|^2_{V_{\mathcal{O}_R}} := (u|u)_{V_{\mathcal{O}_R}}$ for $u \in V_{\mathcal{O}_R}$. The symbols $H'_{\mathcal{O}_R}$ and $V'_{\mathcal{O}_R}$ will stand for the corresponding dual spaces.

Since the sets $\mathcal{O}_R$ are bounded,

$$\textit{the embeddings } V_{\mathcal{O}_R} \hookrightarrow H_{\mathcal{O}_R} \textit{ are compact.} \tag{23}$$

Let $q \in (1, \infty)$. Let us consider the following three functional spaces, analogous to those considered in [25] and [9], see also [22, 23]:

$\mathbb{D}([0,T],U')$:= the space of càdlàg functions $u:[0,T] \to U'$ with the topology $\mathscr{T}_1$ induced by the Skorokhod metric δ_T,

$L^q_w(0,T;V)$:= the space $L^q(0,T;V)$ with the weak topology $\mathscr{T}_2$,

$L^q(0,T;H_{\mathrm{loc}})$:= the space of measurable functions $u:[0,T] \to H$ such that for all $R \in \mathbb{N}$

$$p_{T,R}\,(u) := \|u\|_{L^q(0,T;H_{\mathscr{O}_R})} := \left(\int_0^T \int_{\mathscr{O}_R} |u(t,x)|^q dxdt \right)^{\frac{1}{q}} < \infty, \tag{24}$$

with the topology $\mathscr{T}_3$ generated by the seminorms $(p_{T,R})_{R\in\mathbb{N}}$.

Let H_w denote the Hilbert space H endowed with the weak topology. Let us consider the fourth space, see [25],

$\mathbb{D}([0,T];H_w)$:= the space of weakly càdlàg functions $u:[0,T] \to H$ with the weakest topology $\mathscr{T}_4$ such that for all $h \in h$ the mappings

$$\mathbb{D}([0,T];\ H_w) \ni u \mapsto (u(\cdot)|h)_H \in \mathbb{D}([0,T];\mathbb{R}) \text{ are continuous.} \tag{25}$$

In particular, $u_n \to u$ in $\mathbb{D}([0,T];H_w)$ iff for all $h \in H$:

$$(u_n(\cdot)|h)_H \to (u(\cdot)|h)_H \quad \text{in the space} \quad \mathbb{D}([0,T];\mathbb{R}).$$

Let us consider the ball

$$\mathbb{B} := \{x \in H: \ \ |x|_H \le r\}.$$

Let $\mathbb{B}_w$ denote the ball $\mathbb{B}$ endowed with the weak topology. It is well-known that the $\mathbb{B}_w$ is metrizable, see [5]. Let q_r denote the metric compatible with the weak topology on $\mathbb{B}$. Let us consider the following space

$$\mathbb{D}([0,T];\mathbb{B}_w) = \text{ the space of weakly càdlàg functions } u:[0,T]\to H \text{ and such that } \sup_{t\in[0,T]} |u(t)|_H \le r. \tag{26}$$

Then $\mathbb{D}([0,T];\mathbb{B}_w)$ is metrizable with

$$\delta_{T,r}(u,v) = \inf_{\lambda\in\Lambda_T} \left\{ \sup_{t\in[0,T]} q_r(u(t), v\circ\lambda(t)) + \sup_{t\in[0,T]} |t-\lambda(t)| + \sup_{s\neq t} \left| \log \frac{\lambda(t)-\lambda(s)}{t-s} \right| \right\}. \tag{27}$$

Since by the Banach–Alaoglu Theorem $\mathbb{B}_w$ is compact, $(\mathbb{D}([0,T];\mathbb{B}_w), \delta_{T,r})$ is a complete metric space.

The following lemma says that any sequence $(u_n) \subset L^\infty(0,T;H)$ convergent in $\mathbb{D}([0,T];U')$ is also convergent in the space $\mathbb{D}([0,T];\mathbb{B}_w)$.

Lemma 2 (see Lemma 4.3 in [25]) *Let* $u_n : [0,T] \to H$, $n \in \mathbb{N}$, *be functions such that*

(i) $\sup_{n\in\mathbb{N}} \sup_{s\in[0,T]} |u_n(s)|_H \le r$,
(ii) $u_n \to u$ *in* $\mathbb{D}([0,T];U')$.

Then $u, u_n \in \mathbb{D}([0,T];\mathbb{B}_w)$ *and* $u_n \to u$ *in* $\mathbb{D}([0,T];\mathbb{B}_w)$ *as* $n \to \infty$.

We recall the proof in Appendix E.

The following Theorem is a generalization of the results of [9] and [25]. In the paper [25] the analogous result is proved in the case when the embedding $V \subset H$ is dense and compact. In [9] the embedding $V \subset H$ is only dense and continuous. However, instead of the spaces of càdlàg functions, appropriate spaces of continuous functions are used. The following result generalizes both [9] and [25] in the sense that the embedding $V \subset H$ is dense and continuous and appropriate spaces of càdlàg functions are considered, i.e. $\mathbb{D}([0,T];U')$ and $\mathbb{D}([0,T],H_w)$.

Theorem 2 *Let* $q \in (1,\infty)$ *and let*

$$\mathscr{Z}_q := L^q_w(0,T;V) \cap L^q(0,T;H_{\mathrm{loc}}) \cap \mathbb{D}([0,T];U') \cap \mathbb{D}([0,T],H_w) \tag{28}$$

and let $\mathscr{T}$ *be the supremum of the corresponding topologies. Then a set* $\mathscr{K} \subset \mathscr{Z}_q$ *is* $\mathscr{T}$*-relatively compact if the following three conditions hold*

(a) *for all* $u \in \mathscr{K}$ *and all* $t \in [0,T]$, $u(t) \in H$ *and* $\sup_{u\in\mathscr{K}} \sup_{s\in[0,T]} |u(s)|_H < \infty$,
(b) $\sup_{u\in\mathscr{K}} \int_0^T \|u(s)\|_V^q \, ds < \infty$, *i.e.* $\mathscr{K}$ *is bounded in* $L^q(0,T;V)$,
(c) $\lim_{\delta\to 0} \sup_{u\in\mathscr{K}} w_{[0,T],U'}(u;\delta) = 0$.

Proof We can assume that $\mathscr{K}$ is a closed subset of $\mathscr{Z}_q$. Because of the assumption (b), the weak topology in $L^q_w(0,T;V)$ induced on $\mathscr{Z}_q$ is metrizable. Since the topology in $L^q(0,T;H_{\mathrm{loc}})$ is defined by the countable family of seminorms (24), this space is also metrizable. By assumption (a), it is sufficient to consider the metric subspace $\mathbb{D}([0,T];\mathbb{B}_w) \subset \mathbb{D}([0,T],H_w)$ defined by Eqs. 26 and 27 with $r := \sup_{u\in\mathscr{K}} \sup_{s\in[0,T]} |u(s)|_H$. Thus compactness of a subset of $\mathscr{Z}_q$ is equivalent to its sequential compactness. Let (u_n) be a sequence in $\mathscr{K}$. By the Banach–Alaoglu Theorem condition (b) yields that the set $\mathscr{K}$ is compact in $L^q_w(0,T;V)$.

Using the compactness criterion in the space of càdlàg functions contained in Theorem 1, we will prove that (u_n) is compact in $\mathbb{D}([0,T];U')$. Indeed, by (a) for every $t \in [0,T]$ the set $\{u_n(t), n \in \mathbb{N}\}$ is bounded in H. Since the embedding $H \subset U'$ is compact, the set $\{u_n(t), n \in \mathbb{N}\}$ is compact in U'. This together with condition (c) implies compactness of the sequence (u_n) in the space $\mathbb{D}([0,T];U')$.

Therefore there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that

$$u_{n_k} \to u \quad \text{in} \quad L^q_w(0,T;V) \cap \mathbb{D}([0,T];U') \quad \text{as} \quad k \to \infty.$$

Since $u_{n_k} \to u$ in $\mathbb{D}([0,T];U')$, $u_{n_k}(t) \to u(t)$ in U' for all continuity points of function u, (see [4]). By condition (a) and the Lebesgue dominated convergence theorem, we infer that for all $p \in [1,\infty)$

$$u_{n_k} \to u \quad \text{in} \quad L^p(0,T;U') \quad \text{as} \quad k \to \infty.$$

We claim that

$$u_{n_k} \to u \quad \text{in} \quad L^q(0,T;H_{\mathrm{loc}}) \quad \text{as} \quad k \to \infty.$$

In order to prove it let us fix $R > 0$. Since, by Eq. 23 the embedding $V_{\mathcal{O}_R} \hookrightarrow H_{\mathcal{O}_R}$ is compact and the embeddings $H_{\mathcal{O}_R} \hookrightarrow H' \hookrightarrow U'$ are continuous, by the Lions Lemma, [20], for every $\varepsilon > 0$ there exists a costant $C = C_{\varepsilon,R} > 0$ such that

$$|u|^q_{H_{\mathcal{O}_R}} \le \varepsilon \|u\|^q_{V_{\mathcal{O}_R}} + C_\varepsilon |u|^q_{U'}, \qquad u \in V.$$

Thus for almost all $s \in [0,T]$

$$|u_{n_k}(s) - u(s)|^q_{H_{\mathcal{O}_R}} \le \varepsilon \|u_{n_k}(s) - u(s)\|^q_{V_{\mathcal{O}_R}} + C_\varepsilon |u_{n_k}(s) - u(s)|^q_{U'}, \quad k \in \mathbb{N},$$

and so for all $k \in \mathbb{N}$

$$\|u_{n_k} - u\|^q_{L^q(0,T;H_{\mathcal{O}_R})} \le \varepsilon \|u_{n_k} - u\|^q_{L^q(0,T;V_{\mathcal{O}_R})} + C_\varepsilon \|u_{n_k} - u\|^q_{L^q(0,T;U')}.$$

Passing to the upper limit as $k \to \infty$ in the above inequality and using the estimate

$$\|u_{n_k} - u\|^q_{L^q(0,T;V_{\mathcal{O}_R})} \le q\big(\|u_{n_k}\|^q_{L^q(0,T;V_{\mathcal{O}_R})} + \|u\|^q_{L^q(0,T;V_{\mathcal{O}_R})}\big) \le 2qc_q,$$

where $c_q = \sup_{u \in \mathcal{K}} \|u\|^q_{L^q(0,T;V)}$, we infer that

$$\limsup_{k\to\infty} \|u_{n_k} - u\|^q_{L^q(0,T;H_{\mathcal{O}_R})} \le 2qc_q\varepsilon,$$

By the arbitrariness of ε,

$$\lim_{k\to\infty} \|u_{n_k} - u\|^q_{L^q(0,T;H_{\mathcal{O}_R})} = 0.$$

The proof of Theorem is thus complete. □

3.3 Tightness Criterion

Let us recall that U, V, H are separable Hilbert spaces such that

$$U \hookrightarrow V \hookrightarrow H,$$

where the embedding $U \hookrightarrow V$ is compact and $V \hookrightarrow H$ is continuous. Using the compactness criterion formulated in Theorem 2 we obtain the corresponding tightness criterion in the space $\mathcal{Z}_q$. Let us first recall that the space $\mathcal{Z}_q$ is defined by

$$\mathcal{Z}_q := L^q_w(0,T;V) \cap L^q(0,T;H_{\mathrm{loc}}) \cap \mathbb{D}([0,T];U') \cap \mathbb{D}([0,T],H_w)$$

and it is equipped with the topology $\mathcal{T}$, see Eq. 28.

Corollary 1 (Tightness Criterion) *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg $\mathbb{F}$-adapted U'-valued processes such that*

(a) *there exists a positive constant C_1 such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{s\in[0,T]} |X_n(s)|_H\right] \le C_1,$$

(b) *there exists a positive constant C_2 such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|X_n(s)\| Vq\, ds\right] \le C_2,$$

(c) $(X_n)_{n\in\mathbb{N}}$ *satisfies the Aldous condition* **[A]** in U'.

Let $\tilde{\mathbb{P}}_n$ *be the law of* X_n *on* $\mathscr{Z}_q$. *Then for every* $\varepsilon > 0$ *there exists a compact subset* K_ε *of* $\mathscr{Z}_q$ *such that*

$$\tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

We recall the Aldous condition **[A]** in Appendix A, see Definition 5. The proof of Corrollary 1 is postponed to Appendix A, as well.

4 Stochastic Navier–Stokes Equations Driven by Lévy Noise

4.1 Time Homogeneous Poisson Random Measure

We follow the approach due to Brzeźniak and Hausenblas [6, 7], see also [17] and [26]. Let us denote $\mathbb{N} := \{0, 1, 2, ...\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\mathbb{R}_+ := [0, \infty)$. Let $(S, \mathscr{S})$ be a measurable space and let $M_{\overline{\mathbb{N}}}(S)$ be the set of all $\overline{\mathbb{N}}$ valued measures on $(S, \mathscr{S})$. On the set $M_{\overline{\mathbb{N}}}(S)$ we consider the σ-field $\mathscr{M}_{\overline{\mathbb{N}}}(S)$ defined as the smallest σ-field such that for all $B \in \mathscr{S}$: the map

$$i_B : M_{\overline{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \overline{\mathbb{N}}$$

is measurable.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual hypotheses, see [21].

Definition 2 (see Appendix C in [7]) Let $(Y, \mathscr{Y})$ be a measurable space. A **time homogeneous Poisson random measure** η on $(Y, \mathscr{Y})$ over $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ is a measurable function

$$\eta : (\Omega, \mathscr{F}) \to \big(M_{\overline{\mathbb{N}}}(\mathbb{R}_+ \times Y), \mathscr{M}_{\overline{\mathbb{N}}}(\mathbb{R}_+ \times Y)\big)$$

such that

(1) for all $B \in \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{Y}$, $\eta(B) := i_B \circ \eta : \Omega \to \overline{\mathbb{N}}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(B)]$;

(2) η is independently scattered, i.e. if the sets $B_j \in \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{Y}$, $j = 1, ..., n$, are disjoint then the random variables $\eta(B_j)$, $j = 1, ..., n$, are independent;

(3) for all $U \in \mathscr{Y}$ the $\overline{\mathbb{N}}$-valued process $\big(N(t, U)\big)_{t\geq 0}$ defined by

$$N(t, U) := \eta((0, t] \times U), \qquad t \geq 0$$

is $\mathbb{F}$-adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of $\mathscr{F}_s$.

If η is a time homogeneous Poisson random measure then the formula

$$\nu(A) := \mathbb{E}[\eta((0, 1] \times A)], \qquad A \in \mathscr{Y}$$

defines a measure on $(Y, \mathscr{Y})$ called an **intensity measure** of η. Moreover, for all $T < \infty$ and all $A \in \mathscr{Y}$ such that $\mathbb{E}\big[\eta((0,T] \times A)\big] < \infty$, the $\mathbb{R}$-valued process $\{\tilde{N}(t,A)\}_{t \in (0,T]}$ defined by

$$\tilde{N}(t,A) := \eta((0,t] \times A) - t\nu(A), \qquad t \in (0,T],$$

is an integrable martingale on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. The random measure $l \otimes \nu$ on $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{Y}$, where l stands for the Lebesgue measure, is called an **compensator** of η and the difference between a time homogeneous Poisson random measure η and its compensator, i.e.

$$\tilde{\eta} := \eta - l \otimes \nu,$$

is called a **compensated time homogeneous Poisson random measure.**

Let us also recall basic properties of the stochastic integral with respect to $\tilde{\eta}$, see [7, 17] and [26] for details. Let $\mathbb{H}$ be a separable Hilbert space and let $\mathscr{P}$ be a predictable σ-field on $[0,T] \times \Omega$. Let $\mathfrak{L}^2_{\nu,T}(\mathscr{P} \otimes \mathscr{Y}, l \otimes \mathbb{P} \otimes \nu; \mathbb{H})$ be a space of all $\mathbb{H}$-valued, $\mathscr{P} \otimes \mathscr{Y}$-measurable processes such that

$$\mathbb{E}\left[\int_0^T \int_Y \|\xi(s,\cdot,y)\|^2_{\mathbb{H}}\, ds d\nu(y)\right] < \infty.$$

If $\xi \in \mathfrak{L}^2_{\nu,T}(\mathscr{P} \otimes \mathscr{Y}, l \otimes \mathbb{P} \otimes \nu; \mathbb{H})$ then the integral process $\int_0^t \int_Y \xi(s,\cdot,y)\, \tilde{\eta}(ds,dy)$, $t \in [0,T]$, is a *càdlàg* L^2-integrable martingale. Moreover, the following isometry formula holds

$$\mathbb{E}\left[\left\|\int_0^t \int_Y \xi(s,\cdot,y)\tilde{\eta}(ds,dy)\right\|^2_{\mathbb{H}}\right] = \mathbb{E}\left[\int_0^t \int_Y \|\xi(s,\cdot,y)\|^2_{\mathbb{H}} ds d\nu(y)\right], \;\; t \in [0,T]. \quad (29)$$

4.2 Statement of the Problem

Problem (1–3) can be written as the following stochastic evolution equation

$$\begin{aligned} du(t) + \big[\mathscr{A}u(t) + B\big(u(t)\big)\big]\, dt &= f(t)\, dt + \int_Y F(t,u(t);y)\tilde{\eta}(dt,dy) \\ &\quad + G(t,u(t))\, dW(t), \qquad t \in [0,T], \\ u(0) &= u_0. \end{aligned} \quad (30)$$

Assumption We assume that

(A.1) $u_0 \in H$, $f \in L^2([0,T];V')$,
(F.1) $\tilde{\eta}$ is a compensated time homogeneous Poisson random measure on a measurable space $(Y, \mathscr{Y})$ over $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with a σ-finite intensity measure ν,
(F.2) $F : [0,T] \times H \times Y \to H$ is a measurable function such that $\int_Y 1\!\!1_{\{0\}}(F(t,x;y))\nu(dy) = 0$ for all $x \in H$ and $t \in [0,T]$. Moreover, there exists a constant L such that

$$\int_Y |F(t,u_1;y) - F(t,u_2;y)|^2_H \nu(dy) \le L|u_1 - u_2|^2_H, \quad u_1, u_2 \in H, \;\; t \in [0,T], \quad (31)$$

and for each $p \in \{2, 4, 4+\gamma, 8+2\gamma\}$ there exists a constant C_p such that

$$\int_Y |F(t,u;y)|_H^p \, \nu(dy) \le C_p(1+|u|_H^p), \qquad u \in H, \quad t \in [0,T], \tag{32}$$

where $\gamma > 0$ is some positive constant.

(F.3) Moreover, for all $v \in \mathscr{V}$ the mapping $\tilde{F}_v$ defined by

$$\big(\tilde{F}_v(u)\big)(t,y) := (F(t,u(t^-);y)|v)_H, \quad u \in L^2(0,T;H), \quad (t,y) \in [0,T]\times Y \tag{33}$$

is a continuous from $L^2(0,T;H)$ into $L^2([0,T]\times Y, dl \otimes \nu; \mathbb{R})$ if in the space $L^2(0,T;H)$ we consider the Fréchet topology inherited from the space $L^2(0,T;H_{\mathrm{loc}})$.[1]

(G.1) $W(t)$ is a cylindrical Wiener process in a separable Hilbert space Y_W defined on the stochastic basis $\big(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}\big)$;

(G.2) $G : [0,T]\times V \to \mathscr{L}_{\mathrm{HS}}(Y_W, H)$ and there exists a constant $L_G > 0$ such that

$$\|G(t,u_1) - G(t,u_2)\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_W,H)} \le L_G \|u_1 - u_2\|_V^2, \quad u_1, u_2 \in V, \ t \in [0,T]. \tag{34}$$

Moreover there exist $\lambda, \kappa \in \mathbb{R}$ and $a \in \big(2 - \frac{2}{3+\gamma}, 2\big]$ such that

$$2\langle \mathscr{A}u|u\rangle - \|G(t,u)\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_W,H)} \ge a\|u\|2 - \lambda|u|_H^2 - \kappa, \quad u \in V, \ t \in [0,T]. \tag{35}$$

(G.3) Moreover, G extends to a continuous mapping $G : [0,T]\times H \to \mathscr{L}_{\mathrm{HS}}(Y_W, V')$ such that

$$\|G(t,u)\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_W,V')} \le C(1+|u|_H^2), \qquad u \in H. \tag{36}$$

for some $C > 0$. Moreover, for every $v \in \mathscr{V}$ the mapping $\tilde{G}_v$ defined by

$$\big(\tilde{G}_v(u)\big)(t) := (G(t,u(t))|v)_H, \qquad u \in L^2(0,T;H), \quad t \in [0,T] \tag{37}$$

is a continuous mapping from $L^2(0,T;H)$ into $L^2([0,T]; \mathscr{L}_{\mathrm{HS}}(Y_W, \mathbb{R}))$ if in the space $L^2(0,T;H)$ we consider the Fréchet topology inherited from the space $L^2(0,T;H_{\mathrm{loc}})$.

Let us recall that the space $L^2(0,T;H_{\mathrm{loc}})$ is defined by Eq. 24. For any Hilbert space E the symbol $\mathscr{L}_{\mathrm{HS}}(Y_W;E)$ denotes the space of Hilbert-Schmidt operators from Y_W into E.

Definition 3 **A martingale solution** of Eq. 30 is a system $\big(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{u}, \bar{\eta}, \bar{W}\big)$, where

- $\big(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}\big)$ is a filtered probability space with a filtration $\bar{\mathbb{F}} = \{\bar{\mathscr{F}}_t\}_{t\ge 0}$,
- $\bar{\eta}$ is a time homogeneous Poisson random measure on $(Y, \mathscr{Y})$ over $\big(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}\big)$ with the intensity measure ν,
- $\bar{W}$ is a cylindrical Wiener process on the space Y_W over $\big(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}\big)$,

[1] Here l denotes the Lebesgue measure on the interval $[0,T]$.

- $\bar{u} : [0,T] \times \Omega \to H$ is a predictable process with $\bar{\mathbb{P}}$-a.e. paths

$$\bar{u}(\cdot,\omega) \in \mathbb{D}\big([0,T], H_w\big) \cap L^2(0,T;V)$$

such that for all $t \in [0,T]$ and all $v \in V$ the following identity holds $\bar{\mathbb{P}}$-a.s.

$$\begin{aligned}(\bar{u}(t)v)_H \; &+ \int_0^t \langle \mathscr{A}\bar{u}(s)|v\rangle ds + \int_0^t \langle B(\bar{u}(s))|v\rangle ds \\ = (u_0|v)_H &+ \int_0^t \langle f(s)|v\rangle ds + \int_0^t \int_Y (F(s,\bar{u}(s);y)|v)_H \, \tilde{\bar{\eta}}(ds,dy) \\ &+ \Big\langle \int_0^t G(s,\bar{u}(s))\, d\bar{W}(s) v \Big\rangle .\end{aligned}$$

We will prove existence of a martingale solution of the Eq. 30. To this end we use the Faedo–Galerkin method. The Galerkin approximations generate a sequence of probability measures on appropriate functional space. We will prove that this sequence is tight. Let us emphasize that to prove the tightness, assumption (F.2) with $p=2$ in inequality (32) is sufficient. The stronger condition on p, i.e. inequality (32) for a certain $p>4$, is connected with the construction of the process $\bar{u}$ to deal with the nonlinear term. Assumptions (G.2)–(G.3) allow to consider the Gaussian noise term G dependent both on u and ∇u. This corresponds to inequality (35) with $a<2$. The case when $a=2$ is related to the noise term G dependent on u but not on its gradient. Moreover, assumptions (F.3) and (G.3) are important in the case of unbounded domain $\mathscr{O}$. In the case when $\mathscr{O}$ is bounded, they can be omitted, see [25].

5 Existence of Solutions

Theorem 3 *There exists a martingale solution of the problem* (30) *provided assumptions* (A.1), (F.1)–(F.3) *and* (G.1)–(G.3) *are satisfied.*

5.1 Faedo–Galerkin Approximation

Let $\{e_i\}_{i=1}^{\infty}$ be the orthonormal basis in H composed of eigenvectors of the operator L defined by Eq. 18. Let $H_n := span\{e_1, ..., e_n\}$ be the subspace with the norm inherited from H and let $P_n : H \to H_n$ be defined by Eq. 20. Let us fix $m > \frac{d}{2}+1$ and let U be the space defined by Eq. 16. Consider the following mapping

$$B_n(u) := P_n B(\chi_n(u), u), \qquad u \in H_n,$$

where $\chi_n : H \to H$ is defined by $\chi_n(u) = \theta_n(|u|_{U'})u$, where $\theta_n : \mathbb{R} \to [0,1]$ of class $\mathscr{C}^{\infty}$ such that

$$\theta_n(r) = 1 \quad \text{if} \quad r \le n \qquad \text{and} \qquad \theta_n(r) = 0 \quad \text{if} \quad r \ge n+1.$$

Since $H_n \subset H$, B_n is well defined. Moreover, $B_n : H_n \to H_n$ is globally Lipschitz continuous.

Let us consider the classical Faedo–Galerkin approximation in the space H_n

$$
\begin{aligned}
u_n(t) = P_n u_0 - \int_0^t \big[P_n \mathscr{A} u_n(s) + B_n\big(u_n(s)\big) - P_n f(s) \big]\, ds \\
+ \int_0^t \int_Y P_n F(s, u_n(s^-), y) \tilde{\eta}(ds, dy) \\
+ \int_0^t P_n G(s, u_n(s))\, dW(s), \quad t \in [0, T].
\end{aligned} \tag{38}
$$

Lemma 3 *For each $n \in \mathbb{N}$, there exists a unique $\mathbb{F}$-adapted, càdlàg H_n valued process u_n satisfying the Galerkin* (38).

Proof The assertion follows from Theorem 9.1 in [17]. □

Using the Itô formula, see [17] or [21], and the Burkholder–Davis–Gundy inequality, see [26], we will prove the following lemma about *a priori* estimates of the solutions u_n of Eq. 38. In fact, these estimates hold provided the noise terms satisfy only condition (32) in assumption (F.2) and condition (35) in assumption (G.2).

Lemma 4 *The processes $(u_n)_{n \in \mathbb{N}}$ satisfy the following estimates.*

(i) *For every $p \in [1, 4+\gamma]$ there exists a positive constant $C_1(p)$ such that*

$$
\sup_{n \geq 1} \mathbb{E} \left(\sup_{0 \leq s \leq T} |u_n(s)|_H^p \right) \leq C_1(p). \tag{39}
$$

(ii) *There exists a positive constant C_2 such that*

$$
\sup_{n \geq 1} \mathbb{E} \left[\int_0^T \|u_n(s)\|_V^2\, ds \right] \leq C_2. \tag{40}
$$

Let us recall that $\gamma > 0$ is defined in assumption (F.2).

Proof For all $n \in \mathbb{N}$ and all $R > 0$ let us define

$$
\tau_n(R) := \inf\{t \geq 0 : \ |u_n(t)|_H \geq R\} \wedge T. \tag{41}
$$

Since the process $\big(u_n(t)\big)_{t \in [0,T]}$ is $\mathbb{F}$-adapted and right-continuous, $\tau_n(R)$ is a stopping time. Moreover, since the process (u_n) is càdlàg on $[0, T]$, the trajectories $t \mapsto u_n(t)$ are bounded on $[0, T]$, $\mathbb{P}$-a.s. Thus $\tau_n(R) \uparrow T$, $\mathbb{P}$-a.s., as $R \uparrow \infty$.

Assume first that $p = 2$ or $p = 4 + \gamma$. Using the Itô formula to the function $\phi(x) := |x|^p := |x|_H^p$, $x \in H$, we obtain for all $t \in [0, T]$

$$
\begin{aligned}
\big|u_n\ (t \wedge \tau_n(R))\big|^p &= |P_n u_0|^p \\
&+ \int_0^{t\wedge\tau_n(R)} \big\{p\big|u_n(s)\big|^{p-2}\langle u_n(s) - P_n\mathscr{A}u_n(s) - B_n(u_n(s)) + P_n f(s)\rangle\big\}\,ds \\
&+ \int_0^{t\wedge\tau_n(R)} \int_Y \big\{\phi\big(u_n(s^-) + P_nF(s, u_n(s^-); y)\big) - \phi\big(u_n(s^-)\big)\big\}\,\tilde{\eta}(ds, dy) \\
&+ \int_0^{t\wedge\tau_n(R)} \int_Y \big\{\phi\big(u_n(s^-) + P_nF(s, u_n(s^-); y)\big) - \phi\big(u_n(s^-)\big) \\
&\qquad - \langle \phi'(u_n(s^-)) | P_nF(s, u_n(s^-); y)\rangle\big\}\,\nu(ds, dy) \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_n(R)} \mathrm{Tr}\left[P_nG(s, u_n(s))\frac{\partial^2\phi}{\partial x^2}\big(P_nG(s, u_n(s))\big)^*\right] ds \\
&+ \int_0^{t\wedge\tau_n(R)} p\,|u_n(s)|^{p-2}\langle u_n(s)\,P_nG(s, u_n(s))\,dW(s)\rangle.
\end{aligned}
$$

By Eqs. 11 and 15 we obtain for all $t \in [0, T]$

$$
\begin{aligned}
\big|u_n\ (t \wedge \tau_n(R))\big|^p &= |P_n u_0|^p \\
&+ \int_0^{t\wedge\tau_n(R)} \big\{-p\big|u_n(s)\big|^{p-2}\|u_n(s)\|^2 + p\big|u_n(s)\big|^{p-2}\langle u_n(s)\,f(s)\rangle\big\}\,ds \\
&+ \int_0^{t\wedge\tau_n(R)} \int_Y \big\{\big|u_n(s^-) + P_nF(s, u_n(s^-); y)\big|^p - \big|u_n(s^-)\big|^p\big\}\,\tilde{\eta}(ds, dy) \\
&+ \int_0^{t\wedge\tau_n(R)} \int_Y \big\{\big|u_n(s^-) + P_nF(s, u_n(s^-); y)\big|^p - \big|u_n(s^-)\big|^p \\
&\qquad - p\big|u_n(s^-)\big|^{p-2}\langle u_n(s^-)\,P_nF(s, u_n(s^-); y)\rangle\big\}\,\nu(dy)ds \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_n(R)} \mathrm{Tr}\left[P_nG(s, u_n(s))\frac{\partial^2\phi}{\partial x^2}\big(P_nG(s, u_n(s))\big)^*\right] ds \\
&+ \int_0^{t\wedge\tau_n(R)} p\,|u_n(s)|^{p-2}\langle u_n(s)G(s, u_n(s))\,dW(s)\rangle.
\end{aligned}
$$

Let us recall that according to Eq. 15 we have $\langle \mathscr{A}u|u\rangle = ((u|u))$ and thus

$$
2\langle \mathscr{A}u|u\rangle - a\|u\|^2 = (2-a)\|u\|^2.
$$

Hence inequality (35) in assumption (G.2) can be written equivalently in the following form

$$
\|G(s, u)\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_W, H)} \le (2-a)\|u\|^2 + \lambda|u|_H^2 + \kappa, \qquad u \in V, \quad s \in [0, T].
$$

Hence

$$\frac{1}{2}\int_0^{t\wedge\tau_n(R)} \mathrm{Tr}\left[P_nG(s,u_n(s))\frac{\partial^2\phi}{\partial x^2}\big(P_nG(s,u_n(s))\big)^*\right] ds$$
$$\le \frac{p(p-1)}{2}\int_0^{t\wedge\tau_n(R)} |u_n(s)|^{p-2}\big[(2-a)\|u_n(s)\|^2+\lambda|u_n(s)|^2+\kappa\big]\,ds.$$

Moreover, by assumption (A.1), Eq. 8 and the Schwarz inequality, we obtain for every $\varepsilon>0$ and for all $s\in[0,T]$

$$\langle f(s)|u_n(s)\rangle \le |f(s)|_{V'}\cdot\|u_n(s)\|_V$$
$$\le |f(s)|_{V'}\,|u_n(s)| + \frac{1}{4\varepsilon}|f(s)|^2_{V'} + \varepsilon\,\|u_n(s)\|^2$$

and hence by the Young inequality[2]

$$|P_nu_0|^p + p|u_n(s)|^{p-2}\left(|f(s)|_{V'}|u_n(s)| + \frac{1}{4\varepsilon}|f(s)|^2_{V'}\right)$$
$$+\frac{p(p-1)}{2}|u_n(s)|^{p-2}\big[\lambda|u_n(s)|^2+\kappa\big] = \frac{p(p-1)\lambda}{2}\,|u_n(s)|^p$$
$$+|P_nu_0|^p + p|f(s)|_{V'}|u_n(s)|^{p-1} + p\left(\frac{1}{4\varepsilon}|f(s)|^2_{V'} + \frac{(p-1)\kappa}{2}\right)|u_n(s)|^{p-2}$$
$$\le c + c_1|u_n(s)|^p$$

for some constants $c, c_1>0$. Thus

$$\big|u_n\ (t\wedge\tau_n(R))\big|^p + \left[p - p\varepsilon - \frac{1}{2}p(p-1)(2-a)\right]\int_0^{t\wedge\tau_n(R)}\big|u_n(s)\big|^{p-2}\|u_n(s)\|^2 ds$$
$$\le\ c + c_1\int_0^{t\wedge\tau_n(R)}\big|u_n(s)\big|^p\,ds$$
$$+\int_0^{t\wedge\tau_n(R)}\int_Y \left\{\big|u_n(s^-)+P_nF(s,u_n(s^-);y)\big|^p - \big|u_n(s^-)\big|^p\right\}\tilde\eta(ds,dy)$$
$$+\int_0^{t\wedge\tau_n(R)}\int_Y \Big\{\big|u_n(s^-)+P_nF(s,u_n(s^-);y)\big|^p - \big|u_n(s^-)\big|^p$$
$$- p\big|u_n(s^-)\big|^{p-2}(u_n(s^-)|P_nF(s,u_n(s^-);y)_H\Big\}\,\nu(dy)ds$$
$$+\int_0^{t\wedge\tau_n(R)} p\,|u_n(s)|^{p-2}\langle u_n(s)|G(s,u_n(s))\,dW(s)\rangle. \tag{42}$$

Let us choose $\varepsilon>0$ such that $p - p\varepsilon - \frac{1}{2}p(p-1)(2-a)>0$, or equivalently,

$$\varepsilon < 1 - \frac{1}{2}(p-1)(2-a).$$

Note that since by assumption (G.2) $a\in\big(2-\frac{2}{3+\gamma},2]$, such an ε exists.

[2] $ab\le\frac{1}{q_1}a^{q_1}+\frac{1}{q_2}b^{q_2}$ if $a,b>0$, $q_1,q_2\in(1,\infty)$ and $\frac{1}{q_1}+\frac{1}{q_2}=1$.

From the Taylor formula, it follows that for each $p \geq 2$ there exists a positive constant $c_p > 0$ such that for all $x, h \in H$ the following inequality holds

$$\big| |x+h|_H^p - |x|_H^p - p|x|_H^{p-2}(x|h)_H \big| \leq c_p(|x|_H^{p-2} + |h|_H^{p-2})\, |h|_H^2. \tag{43}$$

By Eqs. 43, 32 and 41, the process $\big(M_n(t \wedge \tau_n(R))\big)_{t\in[0,T]}$, where

$$M_n(t) := \int_0^t \int_Y \big\{ \big|u_n(s^-) + P_n F(s, u_n(s^-); y)\big|^p - \big|u_n(s^-)\big|^p \big\}\, \tilde{\eta}(ds, dy),$$

$t \in [0,T]$, is an integrable martingale. Hence $\mathbb{E}[M_n(t \wedge \tau_n(R))] = 0$ for all $t \in [0,T]$. By Eq. 35 and 41, the process $\big(N_n(t \wedge \tau_n(R))\big)_{t\in[0,T]}$, where

$$N_n(t) := \int_0^t |u_n(s)|^{p-2} \langle u_n(s) | G(s, u_n(s))\, dW(s) \rangle, \qquad t \in [0,T]$$

is an integrable martingale and thus $\mathbb{E}[N_n(t \wedge \tau_n(R))] = 0$ for all $t \in [0,T]$.

Let us denote

$$\begin{aligned} I_n(t) := \int_0^t \int_Y \big\{ &\big|u_n(s^-) + P_n F(s, u_n(s^-); y)\big|^p - \big|u_n(s^-)\big|^p \\ &- p\big|u_n(s^-)\big|^{p-2} (u_n(s^-) | P_n F(s, u_n(s^-); y))_H \big\}\, \nu(dy) ds, \quad t \in [0,T]. \end{aligned} \tag{44}$$

By Eqs. 43 and 32 we obtain the following inequalities

$$\begin{aligned} &\big|I_n(t)\big| \\ &\leq c_p \int_0^t \int_Y \big|P_n F(s, u_n(s^-); y)\big|_H^2 \big\{ \big|u_n(s^-)\big|_H^{p-2} + \big|P_n F(s, u_n(s^-); y)\big|_H^{p-2} \big\} \nu(dy) ds \\ &\leq c_p \int_0^t \big\{ C_2 \big|u_n(s)\big|_H^{p-2} \big(1 + \big|u_n(s)\big|_H^2\big) + C_p\big(1 + \big|u_n(s)\big|_H^p\big) \big\}\, ds \\ &\leq \tilde{c}_p \int_0^t \big\{ 1 + \big|u_n(s)\big|_H^p \big\}\, ds = \tilde{c}_p t + \tilde{c}_p \int_0^t \big|u_n(s)\big|_H^p\, ds, \qquad t \in [0,T] \end{aligned}$$

for some constant $\tilde{c}_p > 0$. Thus by the Fubini Theorem, we obtain the following inequality

$$\mathbb{E}\big[\big|I_n(t)\big|\big] \leq \tilde{c}_p t + \tilde{c}_p \int_0^t \mathbb{E}\big[\big|u_n(s)\big|_H^p\big]\, ds, \qquad t \in [0,T]. \tag{45}$$

By Eqs. 42 and 45, we have for all $t \in [0,T]$

$$\begin{aligned} &\mathbb{E}\big[\big|u_n(t \wedge \tau_n(R))\big|_H^p\big] \\ &+ \Big[p - p\varepsilon - \frac{1}{2}p(p-1)(2-a)\Big] \mathbb{E}\Big[\int_0^{T\wedge\tau_n(R)} \big|u_n(s)\big|_H^{p-2} \|u_n(s)\|^2\, ds\Big] \\ &\leq c + \tilde{c}_p T + (c_1 + \tilde{c}_p) \int_0^{t\wedge\tau_n(R)} \mathbb{E}\big[\big|u_n(s)\big|_H^p\big]\, ds. \end{aligned} \tag{46}$$

In particular,

$$\mathbb{E}\big[\big|u_n(t \wedge \tau_n(R))\big|_H^p\big] \leq c + \tilde{c}_p T + (c_1 + \tilde{c}_p) \int_0^{t\wedge\tau_n(R)} \mathbb{E}\big[\big|u_n(s)\big|_H^p\big]\, ds.$$

By the Gronwall Lemma we infer that for all $t \in [0, T]$: $\mathbb{E}\big[\big|u_n(t \wedge \tau_n(R))\big|^p\big] \le \tilde{\tilde{C}}_p$ for some constant $\tilde{\tilde{C}}_p$ independent of $t \in [0, T]$, $R > 0$ and $n \in \mathbb{N}$, i.e.

$$\sup_{n\ge 1} \sup_{t\in[0,T\wedge\tau_n(R)]} \mathbb{E}\big[\big|u_n(t)\big|_H^p\big] \le \tilde{\tilde{C}}_p.$$

Hence, in particular,

$$\sup_{n\ge 1} \mathbb{E}\left[\int_0^{T\wedge\tau_n(R)} \big|u_n(s)\big|_H^p \, ds\right] \le \tilde{C}_p$$

for some constant $\tilde{C}_p > 0$. Passing to the limit as $R \uparrow \infty$, by the Fatou Lemma we infer that

$$\sup_{n\ge 1} \mathbb{E}\left[\int_0^{T} \big|u_n(s)\big|_H^p \, ds\right] \le \tilde{C}_p. \tag{47}$$

By Eqs. 46 and 47, we infer that

$$\sup_{n\ge 1} \mathbb{E}\left[\int_0^{T\wedge\tau_n(R)} \big|u_n(s)\big|_H^{p-2} \|u_n(s)\|^2 \, ds\right] \le C_p \tag{48}$$

for some positive constant C_p. Passing to the limit as $R \uparrow \infty$ and using again the Fatou Lemma we infer that

$$\sup_{n\ge 1} \mathbb{E}\left[\int_0^{T} \big|u_n(s)\big|_H^{p-2} \|u_n(s)\|^2 \, ds\right] \le C_p. \tag{49}$$

In particular, putting $p := 2$ by Eqs. 8, 49 and 47 we obtain assertion (40).

Let us move to the proof of inequality (39). By the Burkholder–Davis–Gundy inequality we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[0,t]} |M_n(r\wedge\tau_n(R))|\right] \\
&= \mathbb{E}\left[\sup_{r\in[0,t]} \left|\int_0^{r\wedge\tau_n(R)} \int_Y \big\{\big|u_n(s^-) + P_n F(s, u_n(s^-); y)\big|_H^p - \big|u_n(s^-)\big|_H^p\big\} \tilde{\eta}(ds, dy)\right|\right] \\
&\le \tilde{K}_p \mathbb{E}\left[\left(\int_0^{t\wedge\tau_n(R)} \int_Y \big(\big|u_n(s^-) + P_n F(s, u_n(s^-); y)\big|_H^p - \big|u_n(s^-)\big|_H^p\big)^2 \nu(dy) ds\right)^{\frac{1}{2}}\right]
\end{aligned} \tag{50}$$

for some constant $\tilde{K}_p > 0$. By Eq. 43 and the Schwarz inequality we obtain the following inequalities for all $x, h \in H$

$$\begin{aligned}
\big(|x+h|_H^p - |x|_H^p\big)^2 &\le 2\big\{p^2 |x|_H^{2p-2} |h|_H^2 + c_p^2 \big(|x|_H^{p-2} + |h|_H^{p-2}\big)^2 |h|_H^4\big\} \\
&\le 2p^2 |x|_H^{2p-2} |h|_H^2 + 4c_p^2 |x|_H^{2p-4} |h|_H^4 + 4c_p^2 |h|_H^{2p}.
\end{aligned}$$

Hence by inequality (32) in assumption (F.2) we obtain for all $s \in [0, T]$

$$
\begin{aligned}
&\int_Y \left(\left|u_n(s^-) + P_n F(s, u_n(s^-); y)\right|_H^p - \left|u_n(s^-)\right|_H^p\right)^2 \nu(dy) \\
&\le 2p^2 \left|u_n(s^-)\right|_H^{2p-2} \int_Y \left|F(s, u_n(s^-); y)\right|_H^2 \nu(dy) \\
&+4c_p^2 \left|u_n(s^-)\right|_H^{2p-4} \int_Y \left|F(s, u_n(s^-); y)\right|_H^4 \nu(dy) + 4c_p^2 \int_Y \left|F(s, u_n(s^-); y)\right|_H^{2p} \nu(dy) \\
&\le C_1 + C_2 \left|u_n(s^-)\right|_H^{2p-4} + C_3 \left|u_n(s^-)\right|_H^{2p-2} + C_4 \left|u_n(s^-)\right|_H^{2p} \qquad (51)
\end{aligned}
$$

for some positive constants C_i, $i = 1, ..., 4$. By Eq. 51 and the Young inequality we infer that

$$
\int_Y \left(\left|u_n(s^-) + P_n F(s, u_n(s^-); y)\right|_H^p - \left|u_n(s^-)\right|_H^p\right)^2 \nu(dy) \le K_1 + K_2 \left|u_n(s^-)\right|_H^{2p}
$$

for some positive constants K_1 and K_2. Thus

$$
\begin{aligned}
&\left(\int_0^{t\wedge\tau_n(R)} \int_Y \left(\left|u_n(s^-) + P_n F(s, u_n(s^-); y)\right|_H^p - \left|u_n(s^-)\right|_H^p\right)^2 \nu(dy) ds\right)^{\frac{1}{2}} \\
&\le \sqrt{T K_1} + \sqrt{K_2} \left(\int_0^{t\wedge\tau_n(R)} \left|u_n(s)\right|_H^{2p} ds\right)^{\frac{1}{2}}. \qquad (52)
\end{aligned}
$$

By Eqs. 50, 52 and 47 we obtain the following inequalities

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[0,t]} |M_n(r \wedge \tau_n(R))|\right] \\
&\le \tilde{K}_p \sqrt{T K_1} + \tilde{K}_p \sqrt{K_2} \mathbb{E}\left[\left(\int_0^{t\wedge\tau_n(R)} \left|u_n(s)\right|_H^{2p} ds\right)^{\frac{1}{2}}\right] \\
&\le \tilde{K}_p \sqrt{T K_1} + \tilde{K}_p \sqrt{K_2} \mathbb{E}\left[\left(\sup_{s\in[0,t]} \left|u_n(s \wedge \tau_n(R))\right|_H^p\right)^{\frac{1}{2}} \left(\int_0^{t\wedge\tau_n(R)} \left|u_n(s)\right|_H^p ds\right)^{\frac{1}{2}}\right] \\
&\le \tilde{K}_p \sqrt{T K_1} + \frac{1}{4} \mathbb{E}\left[\sup_{s\in[0,t]} \left|u_n(s \wedge \tau_n(R))\right|_H^p\right] + \tilde{K}_p^2 K_2 \mathbb{E}\left[\int_0^{t\wedge\tau_n(R)} \left|u_n(s)\right|_H^p ds\right] \\
&\le \frac{1}{4} \mathbb{E}\left[\sup_{s\in[0,t]} \left|u_n(s \wedge \tau_n(R))\right|_H^p\right] + \tilde{K}, \qquad (53)
\end{aligned}
$$

where $\tilde{K} = \tilde{K}_p \sqrt{T K_1} + \tilde{K}_p^2 K_2 \tilde{C}_p$. (The constant $\tilde{C}_p$ is the same as in Eq. 47).

Similarly, by the Burkholder–Davis–Gundy inequality we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[0,t\wedge\tau_n(R)]}|N_n(r)|\right]\\
&=\mathbb{E}\left[\sup_{r\in[0,t\wedge\tau_n(R)]}\left|\int_0^r p\,|u_n(s)|^{p-2}\langle u_n(s)\,P_nG(s,u_n(s))\,dW(s)\rangle\right|\right]\\
&\le C\,p\cdot\mathbb{E}\left[\left(\int_0^{t\wedge\tau_n(R)}|u_n(s)|^{2p-2}\cdot\|G(s,u_n(s))^2_{\mathscr{L}_{\mathrm{HS}}(Y,H)}\,ds\right)^{\frac{1}{2}}\right]\\
&\le Cp\mathbb{E}\left[\left(\sup_{s\in[0,t\wedge\tau_n(R)]}|u_n(s)|^p\right)^{\frac{1}{2}}\left(\int_0^{t\wedge\tau_n(R)}|u_n(s)|^{p-2}\cdot\|G(s,u_n(s))\|^2_{\mathscr{L}_{\mathrm{HS}}(Y,H)}ds\right)^{\frac{1}{2}}\right].
\end{aligned}
$$

By inequality (35) in assumption (G.2) and estimates (49), (47) we have the following inequalities

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[0,t]}|N_n(r\wedge\tau_n(R))|\right]\le C\,p\cdot\mathbb{E}\left[\left(\sup_{s\in[0,t]}|u_n(s\wedge\tau_n(R))|^p\right)^{\frac{1}{2}}\right.\\
&\left.\cdot\left(\int_0^{t\wedge\tau_n(R)}|u_n(s)|^{p-2}\cdot\left[\lambda\,|u_n(s)|^2+\kappa+(2-a)\|u_n(s)2\right]\,ds\right)^{\frac{1}{2}}\right]\\
&\le\frac{1}{4}\mathbb{E}\left[\sup_{r\in[0,t]}|u_n(r\wedge\tau_n(R))|^p\right]\\
&+C^2p^2\mathbb{E}\left[\int_0^{t\wedge\tau_n(R)}\left[\lambda|u_n(s)|^p+\kappa|u_n(s)|^{p-2}+(2-a)|u_n(s)|^{p-2}\|u_n(s)2\right]\,ds\right]\\
&\le\frac{1}{4}\mathbb{E}\left[\sup_{r\in[0,t]}|u_n(r\wedge\tau_n(R))|^p\right]+\tilde{\tilde{K}},
\end{aligned}\tag{54}
$$

where $\tilde{\tilde{K}}=C^2p^2[\lambda\tilde{C}_p+\kappa\tilde{C}_{p-2}+(2-a)C_2]$. (The constants $\tilde{C}_p,\tilde{C}_{p-2}$ are the same as in Eq. 47 and C_2 is the same as in Eq. 49.) Therefore by Eq. 42 for all $t\in[0,T]$

$$
\begin{aligned}
&|u_n(t\wedge\tau_n(R))|^p\le c+c_1\int_0^T|u_n(s)|^p\,ds+\sup_{r\in[0,T]}|M_n(r\wedge\tau_n(R))|\\
&+|I_n(T\wedge\tau_n(R))|+\sup_{r\in[0,T]}|N_n(r\wedge\tau_n(R))|,
\end{aligned}\tag{55}
$$

where I_n is defined by Eq. 44. Since inequality (55) holds for all $t \in [0,T]$ and the right-hand side of Eq. 55 is independent of t, we infer that

$$\mathbb{E}\left[\sup_{t\in T\wedge\tau_n(R)} |u_n(t)|^p\right] \le c + c_1\mathbb{E}\left[\int_0^T |u_n(s)|^p\, ds\right]$$
$$+\mathbb{E}\left[\sup_{r\in[0,T\wedge\tau_n(R)]} |M_n(r)|\right] + \mathbb{E}\left[|I_n(T\wedge\tau_n(R))|\right] + \mathbb{E}\left[\sup_{r\in[0,T\wedge\tau_n(R)]} |N_n(r)|\right]. \tag{56}$$

Using inequalities (47), (53), (45) and (54) in Eq. 56 we infer that

$$\mathbb{E}\left[\sup_{t\in T\wedge\tau_n(R)} |u_n(t)|^p\right] \le C_1(p)$$

for some constant $C_1(p)$ independent of $n \in \mathbb{N}$ and $R > 0$. Passing to the limit as $R \to \infty$, we obtain inequality (39). Thus the Lemma holds for $p \in \{2, 4+\gamma\}$.

Let now $p \in [1, 4+\gamma) \setminus \{2\}$. Let us fix $n \in \mathbb{N}$. Then

$$|u_n(t)|_H^p = \left(|u_n(t)|_H^{4+\gamma}\right)^{\frac{p}{4+\gamma}} \le \left(\sup_{t\in[0,T]} |u_n(t)|_H^{4+\gamma}\right)^{\frac{p}{4+\gamma}}, \qquad t \in [0,T].$$

Thus

$$\sup_{t\in[0,T]} |u_n(t)|_H^p \le \left(\sup_{t\in[0,T]} |u_n(t)|_H^{4+\gamma}\right)^{\frac{p}{4+\gamma}}$$

and by the Hőlder inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]} |u_n(t)|_H^p\right] \le \mathbb{E}\left[\left(\sup_{t\in[0,T]} |u_n(t)|_H^{4+\gamma}\right)^{\frac{p}{4+\gamma}}\right]$$
$$\le \left(\mathbb{E}\left[\sup_{t\in[0,T]} |u_n(t)|_H^{4+\gamma}\right]\right)^{\frac{p}{4+\gamma}} \le \left[C_1(4+\gamma)\right]^{\frac{p}{4+\gamma}}.$$

Since $n \in \mathbb{N}$ was chosen in an arbitray way, we infer that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} |u_n(t)|_H^p\right] \le C_1(p),$$

where $C_1(p) = \left[C_1(4+\gamma)\right]^{\frac{p}{4+\gamma}}$. The proof of Lemma is thus complete. □

5.2 Tightness

Let $m > \frac{d}{2} + 1$ be fixed and let U be the space defined by Eq. 16. We will apply Corollary 1 with $q := 2$. So, let us consider the space

$$\mathscr{Z} := L_w^2(0,T;V) \cap L^2(0,T;H_{\mathrm{loc}}) \cap \mathbb{D}([0,T];U') \cap \mathbb{D}([0,T];H_w). \tag{57}$$

For each $n \in \mathbb{N}$, the solution u_n of the Galerkin equation defines a measure $\mathscr{L}(u_n)$ on $(\mathscr{Z}, \mathscr{T})$. Using Corollary 1 we will prove that the set of measures $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ is tight on $(\mathscr{Z}, \mathscr{T})$. The inequalities (39) and (40) in Lemma 4 are of crucial importance. However, to prove tightness it is sufficient to use inequality (39) only with $p = 2$.

Lemma 5 *The set of measures* $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ *is tight on* $(\mathscr{Z}, \mathscr{T})$.

Proof We will apply Corollary 1. By estimates (39) and (40), conditions (a), (b) are satisfied. Thus, it is sufficient to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]** in the space U'. We will use Lemma 9 in Appendix A. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times such that $0 \le \tau_n \le T$. By Eq. 38, we have

$$\begin{aligned} u_n(t) \;&=\; P_n u_0 - \int_0^t P_n \mathscr{A} u_n(s)\, ds - \int_0^t B_n\left(u_n(s)\right)\, ds + \int_0^t P_n f(s)\, ds \\ &\quad + \int_0^t \int_Y P_n F(s, u_n(s^-), y) \tilde{\eta}(ds, dy) + \int_0^t P_n G(s, u_n(s))\, dW(s) \\ &=: J_1^n + J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t) + J_6^n(t), \qquad t \in [0, T]. \end{aligned}$$

Let $\theta > 0$. We will check that each term J_i^n, i=1,...,6, satisfies condition (89) in Lemma 9.

Since $\mathscr{A} : V \to V'$ and $|\mathscr{A}(u)|_{V'} \le \|u$ and the embedding $V' \hookrightarrow U'$ is continuous, by the Hőlder inequality and Eq. 40, we have the following estimates

$$\begin{aligned} &\mathbb{E}\left[\left|J_2^n(\tau_n + \theta) - J_2^n(\tau_n)\right|_{U'}\right] = \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta} P_n \mathscr{A} u_n(s)\, ds\right|_{U'}\right] \\ &\le c\,\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} \left|\mathscr{A} u_n(s)\right|_{V'} ds\right] \le c\,\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} \|u_n(s)\|\, ds\right] \\ &\le c\,\mathbb{E}\left[\theta^{\frac{1}{2}}\left(\int_0^T \|u_n(s)\|^2\, ds\right)^{\frac{1}{2}}\right] \le c\sqrt{C_2}\cdot\theta^{\frac{1}{2}} =: c_2\cdot\theta^{\frac{1}{2}}. \end{aligned}$$

Thus J_2^n satisfies condition (89) with $\alpha = 1$ and $\beta = \frac{1}{2}$.

Let us consider the term J_3^n. Since $m > \frac{d}{2} + 1$ and $U \hookrightarrow V_m$, by Eqs. 12 and 39 we have the following inequalities

$$\begin{aligned} &\mathbb{E}\left[\left|J_3^n(\tau_n + \theta) - J_3^n(\tau_n)\right|_{U'}\right] = \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta} B_n\left(u_n(s)\right)\, ds\right|_{U'}\right] \\ &\le c\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} \left|B\left(u_n(s)\right)\right|_{V_m'} ds\right] \le c\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} \|B\|\cdot|u_n(s)|_H^2\, ds\right] \\ &\le c\|B\|\cdot\mathbb{E}\left[\sup_{s\in[0,T]} |u_n(s)|_H^2\right]\cdot\theta \le c\|B\|\, C_1(2)\cdot\theta =: c_3\cdot\theta, \end{aligned}$$

where $\|B\|$ stands for the norm of $B : H \times H \to V_m'$. This means that J_3^n satisfies condition (89) with $\alpha = \beta = 1$.

Let us move to the term J_4^n. By the Hőlder inequality, we have

$$\mathbb{E}\left[\left|J_4^n(\tau_n+\theta)-J_4^n(\tau_n)\right|_{U'}\right] \le c\mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta} P_n f(s)ds\right|_{V'}\right]$$
$$\le c\cdot\theta^{\frac{1}{2}}\cdot\mathbb{E}\left[\left(\int_0^T |f(s)|_{V'}^2\,ds\right)^{\frac{1}{2}}\right] = c\cdot\theta^{\frac{1}{2}}\cdot\|f\|_{L^2(0,T;V')} =: c_4\cdot\theta^{\frac{1}{2}}.$$

Hence condition (89) holds with $\alpha=1$ and $\beta=\frac{1}{2}$.

Let us consider the term J_5^n. Since $H\hookrightarrow U'$, by Eq. 29, condition (32) with $p=2$ in Assumption (F.2) and by Eq. 39, we obtain the following inequalities

$$\mathbb{E}\left[\left|J_5^n(\tau_n+\theta)-J_5^n(\tau_n)\right|_{U'}^2\right] = \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta}\int_Y P_nF(s,u_n(s);y)\,\tilde{\eta}(ds,dy)\right|_{U'}^2\right]$$
$$\le c\mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta}\int_Y P_nF(s,u_n(s);y)\,\tilde{\eta}(ds,dy)\right|_{H}^2\right]$$
$$= c\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta}\int_Y |P_nF(s,u_n(s);y)|_H^2\,\nu(dy)ds\right] \le C\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta}(1+|u_n(s)|_H^2)ds\right]$$
$$\le C\cdot\theta\cdot\left(1+\mathbb{E}\left[\sup_{s\in[0,T]}|u_n(s)|_H^2\right]\right) \le C\cdot(1+C_1(2))\cdot\theta =: c_5\cdot\theta.$$

Thus J_5^n satisfies condition (89) with $\alpha=2$ and $\beta=1$.

Let us consider the term J_6^n. By the Itô isometry, condition (36) in assumption (G.3), continuity of the embedding $V'\hookrightarrow U'$ and inequality (39), we have

$$\mathbb{E}\left[\left|J_6^n(\tau_n+\theta)-J_6^n(\tau_n)\right|_{U'}^2\right] = \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n+\theta} P_nG(s,u_n(s))\,dW(s)\right|_{U'}^2\right]$$
$$\le c\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta}(1+|u_n(s)|_H^2)ds\right] \le c\theta\left(1+\mathbb{E}\left[\sup_{s\in[0,T]}|u_n(s)|_H^2\right]\right) \le c(1+C_1(2))\theta.$$

Thus J_6^n satisfies condition (89) with $\alpha=2$ and $\beta=1$.

By Lemma 9 the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition in the space U'. This completes the proof of Lemma. □

We will now move to the proof of the main Theorem of existence of a martingale solution. The main difficulties occur in the term containing the nonlinearity B and in the noise terms F and G. To deal with the nonlinear term, we need inequality (39) for some $p>4$. Moreover, we will see that the sequence $(\bar{u}_n)$ of approximate solutions is convergent in the Fréchet space $L^2(0,T;H_{loc})$. So, we will use the property of the mapping B contained in Lemma 6 below. Analogous problems appear in the noise terms, where assumptions (F.3) and (G.3) will be needed in the case when the domain $\mathscr{O}$ is unbounded. For simplicity we assume that $\dim Y_W=1$, i.e. we consider one-dimensional cylindrical Wiener process $W(t)$, $t\in[0,T]$. Construction of a martingale solution is based on the Skorokhod Theorem for nonmetric spaces. The method is closely related to the approach due to Brzeźniak and Hausenblas [6].

5.3 Proof of Theorem 3

By Lemma 5 the set of measures $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ is tight on the space $(\mathscr{Z}, \mathscr{T})$. Let $\eta_n := \eta, n \in \mathbb{N}$. The set of measures $\{\mathscr{L}(\eta_n), n \in \mathbb{N}\}$ is tight on the space $M_{\bar{\mathbb{N}}}([0,T] \times Y)$. Let $W_n := W, n \in \mathbb{N}$. The set $\{\mathscr{L}(W_n), n \in \mathbb{N}\}$ is tight on the space $\mathscr{C}([0,T];\mathbb{R})$ of continuous function from $[0,T]$ to $\mathbb{R}$ with the standard supremum-norm. Thus the set $\{\mathscr{L}(u_n, \eta_n, W_n), n \in \mathbb{N}\}$ is tight on $\mathscr{Z} \times M_{\bar{\mathbb{N}}}([0,T] \times Y) \times \mathscr{C}([0,T];\mathbb{R})$. By Corollary 1 and Remark 2, see Appendix B, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ and, on this space, $\mathscr{Z} \times M_{\bar{\mathbb{N}}}([0,T] \times Y) \times \mathscr{C}([0,T];\mathbb{R})$-valued random variables (u_*, η_*, W_*), $(\bar{u}_k, \bar{\eta}_k, \bar{W}_k)$, $k \in \mathbb{N}$ such that

- (i) $\mathscr{L}\left((\bar{u}_k, \bar{\eta}_k, \bar{W}_k)\right) = \mathscr{L}\left((u_{n_k}, \eta_{n_k}, W_{n_k})\right)$ for all $k \in \mathbb{N}$;
- (ii) $(\bar{u}_k, \bar{\eta}_k, \bar{W}_k) \to (u_*, \eta_*, W_*)$ in $\mathscr{Z} \times M_{\bar{\mathbb{N}}}([0,T] \times Y) \times \mathscr{C}([0,T];\mathbb{R})$ with probability 1 on $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ as $k \to \infty$;
- (iii) $(\bar{\eta}_k(\bar{\omega}), \bar{W}_k(\bar{\omega})) = (\eta_*(\bar{\omega}), W_*(\bar{\omega}))$ for all $\bar{\omega} \in \bar{\Omega}$.

We will denote this sequences again by $((u_n, \eta_n, W_n))_{n\in\mathbb{N}}$ and $((\bar{u}_n, \bar{\eta}_n, \bar{W}_n))_{n\in\mathbb{N}}$. Moreover, $\bar{\eta}_n$, $n \in \mathbb{N}$, and η_* are time homogeneous Poisson random measures on $(Y, \mathscr{Y})$ with intensity measure ν and $\bar{W}_n$, $n \in \mathbb{N}$, and W_* are cylindrical Wiener processes, see [6, Section 9]. Using the definition of the space $\mathscr{Z}$, see Eq. 57, in particular, we have

$$\bar{u}_n \to u_* \quad \text{in} \quad L^2_w(0,T;V) \cap L^2(0,T;H_{\mathrm{loc}}) \cap \mathbb{D}([0,T];U') \cap \mathbb{D}([0,T];H_w) \quad \bar{\mathbb{P}}\text{-a.s.} \tag{58}$$

Since the random variables $\bar{u}_n$ and u_n are identically distributed, we have the following inequalities. For every $p \in [1, 4+\gamma]$

$$\sup_{n\ge 1} \bar{\mathbb{E}}\left(\sup_{0\le s\le T} |\bar{u}_n(s)|_H^p\right) \le C_1(p). \tag{59}$$

and

$$\sup_{n\ge 1} \bar{\mathbb{E}}\left[\int_0^T \|\bar{u}_n(s)\|_V^2\, ds\right] \le C_2. \tag{60}$$

Let us fix $v \in U$. Analogously to [6], let us denote

$$\begin{aligned}
&\mathscr{K}_n(\bar{u}_n, \bar{\eta}_n, \bar{W}_n, v)(t) := (\bar{u}_n(0)|v)_H \\
&+ \int_0^t \langle P_n \mathscr{A} \bar{u}_n(s)|v\rangle ds + \int_0^t \langle B_n(\bar{u}_n(s))|v\rangle ds \\
&+ \int_0^t \langle P_n f(s)|v\rangle ds + \int_0^t \int_Y (P_n F(s, \bar{u}_n(s^-); y)|v)_H\, \tilde{\bar{\eta}}_n(ds, dy) \\
&+ \left\langle \int_0^t P_n G(s, \bar{u}_n(s))\, d\bar{W}_n(s) \Big| v \right\rangle
\end{aligned} \tag{61}$$

and

$$\mathscr{K}(u_*, \eta_*, W_*, v)(t) := (u_*(0)|v)_H + \int_0^t \langle \mathscr{A} u_*(s)|v\rangle ds + \int_0^t \langle B(u_*(s))|v\rangle ds$$
$$+ \int_0^t \langle f(s)|v\rangle\, ds + \int_0^t \int_Y (F(s, u_*(s^-); y)|v)_H\, \tilde{\eta}_*(ds, dy)$$
$$+ \Big\langle \int_0^t G(s, u_*(s))\, dW_*(s) \Big| v \Big\rangle, \quad t \in [0, T]. \tag{62}$$

Step 1^0 We will prove that

$$\lim_{n\to\infty} \|(\bar{u}_n(\cdot)|v)_H - (u_*(\cdot)|v)_H\|_{L^2([0,T]\times\bar{\Omega})} = 0 \tag{63}$$

and

$$\lim_{n\to\infty} \|\mathscr{K}_n(\bar{u}_n, \bar{\eta}_n, \bar{W}_n, v) - \mathscr{K}(u_*, \eta_*, W_*, v)\|_{L^2([0,T]\times\bar{\Omega})} = 0. \tag{64}$$

To prove Eq. 63 let us write

$$\|(\bar{u}_n(\cdot)|v)_H - (u_*(\cdot)|v)_H\|^2_{L^2([0,T]\times\bar{\Omega})}$$
$$= \int_{\bar{\Omega}} \int_0^T |(\bar{u}_n(t) - u_*(t)|v)_H|^2\, dt\bar{\mathbb{P}}(d\omega) = \bar{\mathbb{E}}\left[\int_0^T |(\bar{u}_n(t) - u_*(t)|v)_H|^2\, dt\right].$$

Moreover,

$$\int_0^T |(\bar{u}_n(t) - u_*(t)|v)_H|^2\, dt = \int_0^T \left|\langle \bar{u}_n(t) - u_*(t)|v\rangle_{U',U}\right|^2 dt$$
$$\le \|v\|_U^2 \int_0^T |\bar{u}_n(t) - u_*(t)|^2_{U'}\, dt.$$

Since by Eq. 58 $\bar{u}_n \to u_*$ in $\mathbb{D}([0,T];U')$ and by Eq. 59 $\sup_{t\in[0,T]} |\bar{u}_n(t)|_H^2 < \infty$, $\bar{\mathbb{P}}$-a.s. and the embedding $H \hookrightarrow U'$ is continuous, by the Dominated Convergence Theorem we infer that $\bar{\mathbb{P}}$-a.s., $\bar{u}_n \to u_*$ in $L^2(0,T;U')$. Then

$$\lim_{n\to\infty} \int_0^T |(\bar{u}_n(t) - u_*(t)|v)_H|^2\, dt = 0. \tag{65}$$

Moreover, by the Hőlder inequality and Eq. 59 for every $n \in \mathbb{N}$ and every $r \in \left(1, 2 + \frac{\gamma}{2}\right]$

$$\bar{\mathbb{E}}\left[\left|\int_0^T |\bar{u}_n(t) - u_*(t)|_H^2\, dt\right|^r\right] \le c\bar{\mathbb{E}}\left[\int_0^T \left(|\bar{u}_n(t)|_H^{2r} + |u_*(t)|_H^{2r}\right) dt\right]$$
$$\le \tilde{c}\bar{\mathbb{E}}\left[\sup_{t\in[0,T]} |\bar{u}_n(t)|_H^{2r}\right] \le \tilde{c}C_1(2r) \tag{66}$$

for some constants $c, \tilde{c} > 0$. By Eqs. 66, 65 and the Vitali Theorem we infer that

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^T |(\bar{u}_n(t) - u_*(t)|v)_H|^2 \, dt\right] = 0,$$

i.e. Eq. 63 holds.

Let us move to the proof of Eq. 64. Note that by the Fubini Theorem, we have

$$\begin{aligned}
&\|\mathscr{K}_n(\bar{u}_n, \bar{\eta}_n, \bar{W}_n, v) - \mathscr{K}(u_*, \eta_*, W_*, v)\|^2_{L^2([0,T]\times\bar{\Omega})} \\
&= \int_0^T \int_{\bar{\Omega}} \left|\mathscr{K}_n(\bar{u}_n, \bar{\eta}_n, \bar{W}_n, v)(t) - \mathscr{K}(u_*, \eta_*, W_*, v)(t)\right|^2 d\bar{\mathbb{P}}(\omega) dt \\
&= \int_0^T \bar{\mathbb{E}}\left[\left|\mathscr{K}_n(\bar{u}_n, \bar{\eta}_n, \bar{W}_n, v)(t) - \mathscr{K}(u_*, \eta_*, W_*, v)(t)\right|^2\right] dt.
\end{aligned}$$

We will prove that each term on the right hand side of Eq. 61 tends in $L^2([0,T]\times\bar{\Omega})$ to the corresponding term in Eq. 62.

Since by Eq. 58 $\bar{u}_n \to u_*$ in $\mathbb{D}(0,T;H_w)$ $\bar{\mathbb{P}}$-a.s. and u_* is continuous at $t=0$, we infer that $(\bar{u}_n(0)|v)_H \to (u_*(0)|v)_H$ $\bar{\mathbb{P}}$-a.s. By Eq. 59 and the Vitali Theorem, we have

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[|(\bar{u}_n(0) - u_*(0)|v)_H|^2\right] = 0.$$

Hence

$$\lim_{n\to\infty} \|(\bar{u}_n(0) - u_*(0)|v)_H\|^2_{L^2([0,T]\times\bar{\Omega})} = 0. \tag{67}$$

By Eq. 58 $\bar{u}_n \to u_*$ in $L^2_w(0,T;V)$, $\bar{\mathbb{P}}$-a.s. Moreover, since $v \in U$, $P_n v \to v$ in V, see Section 2.3. Thus by relation (15) we infer that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^t \langle P_n \mathscr{A}\bar{u}_n(s)|v\rangle \, ds &= \lim_{n\to\infty} \int_0^t ((\bar{u}_n(s)|P_n v)) \, ds \\
&= \int_0^t ((u_*(s)\big|v)) \, ds = \int_0^t \big\langle u_*(s)\big|v\big\rangle \, ds.
\end{aligned} \tag{68}$$

By Eq. 15, Lemma 1, the Hőlder inequality and Eq. 59 for all $t \in [0,T]$, $r \in (0, 2+\gamma]$ and $n \in \mathbb{N}$

$$\begin{aligned}
&\bar{\mathbb{E}}\left[\left|\int_0^t \langle P_n \mathscr{A}\bar{u}_n(s)|v\rangle \, ds\right|^{2+r}\right] = \bar{\mathbb{E}}\left[\left|\int_0^t (\bar{u}_n(s)|(A-I)P_n v)_H \, ds\right|^{2+r}\right] \\
&\le c\|v\|_U^{2+r} \bar{\mathbb{E}}\left[\int_0^T |\bar{u}_n(s)|_H^{2+r} \, ds\right] \le \tilde{c}\bar{\mathbb{E}}\left[\sup_{0\le s\le T} |\bar{u}_n(s)|_H^{2+r}\right] \le \tilde{c}C_1(2+r)
\end{aligned} \tag{69}$$

for some constants $c, \tilde{c} > 0$. Therefore by Eqs. 68, 69 and the Vitali Theorem we infer that for all $t \in [0,T]$

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\left|\int_0^t \langle P_n \mathscr{A}\bar{u}_n(s) - \mathscr{A}u_*(s)|v\rangle \, ds\right|^2\right] = 0.$$

Hence by Eq. 59 and the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t \langle P_n \mathscr{A}\bar{u}_n(s) - \mathscr{A}u_*(s)|v\rangle \, ds\right|^2\right] dt = 0. \tag{70}$$

Let us move to the nonlinear term. We will use the following auxiliary result proven in [9]. (We recall the proof in Appendix D.)

Lemma 6 (Lemma B.1 in [9]) *Let $u \in L^2(0,T;H)$ and let $(u_n)_n$ be a bounded sequence in $L^2(0,T;H)$ such that $u_n \to u$ in $L^2(0,T;H_{\mathrm{loc}})$. Let $m > \frac{d}{2}+1$. Then for all $t \in [0,T]$ and all $\psi \in V_m$:*

$$\lim_{n\to\infty} \int_0^t \langle B(u_n(s))|\psi\rangle\, ds = \int_0^t \langle B(u(s))|\psi\rangle\, ds.$$

(*Here $\langle\cdot|\cdot\rangle$ denotes the dual pairing between the space V_m and V_m'.*)

Let us fix $m > \frac{d}{2}+1$. Since by Eqs. 60 and 8 the sequence $(\bar{u}_n)$ is bounded in $L^2(0,T;H)$ and by Eq. 58 $\bar{u}_n \to u_*$ in $L^2(0,T;H_{\mathrm{loc}})$ $\bar{\mathbb{P}}$-a.s., by Lemma 6 we infer that $\bar{\mathbb{P}}$-a.s. for all $t\in[0,T]$ and all $v \in V_m$

$$\lim_{n\to\infty} \int_0^t \langle B(\bar{u}_n(s)) - B(u_*(s))|v\rangle\, ds = 0.$$

It is easy to see that for sufficiently large $n \in \mathbb{N}$

$$B_n(\bar{u}_n(s)) = P_n B(\bar{u}_n(s)), \qquad s \in [0,T].$$

Moreover, if $v \in U$ then $P_n v \to v$ in V_m, see Section 2.3. Since $U \subset V_m$, we infer that for all $v \in U$ and all $t \in [0,T]$

$$\lim_{n\to\infty} \int_0^t \langle B_n(\bar{u}_n(s)) - B(u_*(s))|v\rangle\, ds = 0 \qquad \bar{\mathbb{P}}\text{a.s.} \tag{71}$$

By the Hőlder inequality, Eqs. 12 and 59 we obtain for all $t\in[0,T]$, $r \in \left(0, \frac{\gamma}{2}\right]$ and $n \in \mathbb{N}$

$$\begin{aligned}\bar{\mathbb{E}}\left[\left|\int_0^t \langle B_n(\bar{u}_n(s))|v\rangle\, ds\right|^{2+r}\right] &\le \bar{\mathbb{E}}\left[t^{1+r}\|v\|_{V_m}^{2+r}\int_0^t |B_n(\bar{u}_n(s))|_{V_m'}^{2+r}\, ds\right]\\ \le C\bar{\mathbb{E}}\left[\int_0^t |\bar{u}_n(s)|_H^{2(2+r)}\, ds\right] &\le \tilde{C}\bar{\mathbb{E}}\left[\sup_{s\in[0,T]} |\bar{u}_n(s)|_H^{2(2+r)}\right] \le \tilde{C}C_1(4+2r).\end{aligned} \tag{72}$$

In view of Eqs. 71 and 72, by the Vitali Theorem we obtain for all $t \in [0,T]$

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\left|\int_0^t \langle B_n(\bar{u}_n(s)) - B(u_*(s))|v\rangle\, ds\right|^2\right] = 0. \tag{73}$$

Since by Eq. 59 for all $t\in[0,T]$ and all $n\in\mathbb{N}$

$$\bar{\mathbb{E}}\left[\left|\int_0^t \langle B_n(\bar{u}_n(s))|v\rangle\, ds\right|^2\right] \le c\bar{\mathbb{E}}\left[\sup_{s\in[0,T]} |\bar{u}_n(s)|_H^4\right] \le cC_1(4)$$

for some $c > 0$, by Eq. 73 and the Dominated Convergence Theorem, we infer that

$$\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle B_n(\bar{u}_n(s)) - B(u_*(s)) | v \rangle \, ds \right|^2 \right] dt = 0. \tag{74}$$

Let us move to the noise terms. Let us assume first that $v \in \mathscr{V}$. For all $t \in [0, T]$ we have

$$\begin{aligned}
&\int_0^t \int_Y \left| \langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y) | v \rangle \right|^2 d\nu(y) ds \\
&\int_0^t \int_Y \left| (F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y) | v)_H \right|^2 d\nu(y) ds \\
&= \int_0^t \int_Y \left| \tilde{F}_v(\bar{u}_n)(s, y) - \tilde{F}_v(u_*)(s, y) \right|^2 d\nu(y) ds \\
&\le \int_0^T \int_Y \left| \tilde{F}_v(\bar{u}_n)(s, y) - \tilde{F}_v(u_*)(s, y) \right|^2 d\nu(y) ds \\
&= \| \tilde{F}_v(\bar{u}_n) - \tilde{F}_v(u_*) \|^2_{L^2([0,T]\times Y; \mathbb{R})},
\end{aligned}$$

where $\tilde{F}_v$ is the mapping defined by Eq. 33. Since by Eq. 58 $\bar{u}_n \to u_*$ in $L^2(0, T; H_{\mathrm{loc}})$, $\bar{\mathbb{P}}$-a.s., by assumption (F.3) we infer that for all $t \in [0, T]$

$$\lim_{n\to\infty} \int_0^t \int_Y \left| (F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y) | v)_H \right|^2 d\nu(y) ds = 0. \tag{75}$$

Moreover, by inequality (32) in assumption (F.2) and by Eq. 59 for every $t \in [0, T]$ every $r \in \left(1, 2 + \frac{\gamma}{2}\right]$ and every $n \in \mathbb{N}$ the following inequalities hold

$$\begin{aligned}
&\bar{\mathbb{E}} \left[\left| \int_0^t \int_Y \left| (F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y) | v)_H \right|^2 d\nu(y) ds \right|^r \right] \\
&\le 2^r |v|_H^{2r} \bar{\mathbb{E}} \left[\left| \int_0^t \int_Y \left\{ \left| F(s, \bar{u}_n(s^-); y) \right|_H^2 + \left| F(s, u_*(s^-); y) \right|_H^2 \right\} d\nu(y) ds \right|^r \right] \\
&\le 2^r C_2^r |v|_H^{2r} \, \bar{\mathbb{E}} \left[\left| \int_0^t \left\{ 2 + |\bar{u}_n(s)|_H^2 + |u_*(s)|_H^2 \right\} ds \right|^r \right] \le c \left(1 + \bar{\mathbb{E}} \left[\sup_{s\in[0,T]} |\bar{u}_n(s)|_H^{2r} \right] \right) \\
&\le c(1 + C_1(2r))
\end{aligned} \tag{76}$$

for some constant $c > 0$. Thus by Eqs. 75, 76 and the Vitali Theorem for all $t \in [0, T]$

$$\lim_{n\to\infty} \bar{\mathbb{E}} \left[\int_0^t \int_Y \left| \langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y) | v \rangle \right|^2 d\nu(y) ds \right] = 0, \quad v \in \mathscr{V}. \tag{77}$$

Let now $v \in H$ and let $\varepsilon > 0$. Since $\mathscr{V}$ is dense in H, there exists $v_\varepsilon \in \mathscr{V}$ such that $|v - v_\varepsilon|_H^2 < \varepsilon$. By Eq. 32 the following inequalities hold

$$\begin{aligned}
&\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v\rangle\right|^2 \, d\nu(y)ds \\
&\le 2\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v - v_\varepsilon\rangle\right|^2 \, d\nu(y)ds \\
&\quad +2\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v_\varepsilon\rangle\right|^2 \, d\nu(y)ds \\
&\le 4C_2\varepsilon^2 \int_0^t \left\{2 + |\bar{u}_n(s)|_H^2 + |u_*(s)|_H^2\right\} \, ds \\
&\quad +2\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v_\varepsilon\rangle\right|^2 \, d\nu(y)ds.
\end{aligned}$$

Hence by Eq. 59

$$\begin{aligned}
&\bar{\mathbb{E}}\left[\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v\rangle\right|^2 \, d\nu(y)ds\right] \\
&\le 4C_2\varepsilon^2 \bar{\mathbb{E}}\left[\int_0^t \left\{2 + |\bar{u}_n(s)|_H^2 + |u_*(s)|_H^2\right\} \, ds\right] \\
&\quad +2\bar{\mathbb{E}}\left[\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v_\varepsilon\rangle\right|^2 \, d\nu(y)ds\right] \\
&\le \tilde{c}\varepsilon^2 + 2\bar{\mathbb{E}}\left[\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v_\varepsilon\rangle\right|^2 \, d\nu(y)ds\right].
\end{aligned}$$

Passing to the upper limit as $n \to \infty$ in the above inequality, by Eq. 77 we obtain

$$\limsup_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v\rangle\right|^2 \, d\nu(y)ds\right] \le \tilde{c}\varepsilon^2.$$

Since $\varepsilon > 0$ was chosen in an arbitrary way, we infer that for all $v \in H$

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^t \int_Y \left|\langle F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v\rangle\right|^2 \, d\nu(y)ds\right] = 0.$$

Moreover, since the restriction of P_n to the space H is the $(\cdot|\cdot)_H$-projection onto H_n, see Section 2.3, we infer that also

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^t \int_Y \left|\langle P_n F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v\rangle\right|^2 \, d\nu(y)ds\right] = 0, \quad v \in H.$$

Hence by the properties of the integral with respect to the compensated Poisson random measure and the fact that $\bar{\eta}_n = \eta_*$, we have

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\left|\int_0^t \int_Y \langle P_n F(s, \bar{u}_n(s^-); y) - F(s, u_*(s^-); y)|v\rangle \, \tilde{\eta}_*(ds, dy)\right|^2\right] = 0. \tag{78}$$

Moreover, by inequality (32) in assumption (F.2) and by Eq. 59 we obtain the following inequalities

$$
\begin{aligned}
&\bar{\mathbb{E}}\left[\left|\int_0^t\int_Y \left\langle P_nF(s,\bar{u}_n(s^-);y)-F(s,u_*(s^-);y)|v\right\rangle \tilde{\eta}_*(ds,dy)\right|^2\right]\\
&=\bar{\mathbb{E}}\left[\int_0^t\int_Y \left|(P_nF(s,\bar{u}_n(s^-);y)-F(s,u_*(s^-);y)|v)_H\right|^2 \nu(dy)ds\right]\\
&\le 2|v|_H^2\bar{\mathbb{E}}\left[\int_0^t\int_Y \left\{\left|P_nF(s,\bar{u}_n(s^-);y)\right|_H^2+\left|F(s,u_*(s^-);y)\right|_H^2\right\} \nu(dy)ds\right]\\
&\le 2C_2|v|_H^2\bar{\mathbb{E}}\left[\int_0^t \left\{2+|\bar{u}_n(s)|_H^2+|u_*(s)|_H^2\right\} ds\right]\le c\left(1+\bar{\mathbb{E}}\left[\sup_{s\in[0,T]}|\bar{u}_n(s)|_H^2\right]\right)\\
&\le c(1+C_1(2)).
\end{aligned}
\tag{79}
$$

By Eqs. 78, 79 and the Dominated Convergence Theorem, we have for all $v\in H$

$$
\lim_{n\to\infty}\int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t\int_Y \left\langle P_nF(s,\bar{u}_n(s^-);y)-F(s,u_*(s^-);y)|v\right\rangle \tilde{\eta}_*(ds,dy)\right|^2\right] dt=0.
\tag{80}
$$

Since $U\subset H$, Eq. 80 holds for all $v\in U$, as well.

Let us move to the second part of the noise. Let us assume first that $v\in\mathscr{V}$. We have

$$
\begin{aligned}
&\int_0^t \|\langle G(s,\bar{u}_n(s))-G(s,u_*(s))|v\rangle\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\,ds\\
&=\int_0^t \|(G(s,\bar{u}_n(s))-G(s,u_*(s))|v)_H\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\,ds\\
&=\int_0^t \left\|\tilde{G}_v(\bar{u}_n)(s)-\tilde{G}_v(u_*)(s)\right\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\,ds\\
&\le\int_0^T \left\|\tilde{G}_v(\bar{u}_n)(s)-\tilde{G}_v(u_*)(s)\right\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\,ds\\
&=\|\tilde{G}_v(\bar{u}_n)-\tilde{G}_v(u_*)\|^2_{L^2([0,T];\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R}))},
\end{aligned}
$$

where $\tilde{G}_v$ is the mapping defined by Eq. 37. Since by Eq. 58 $\bar{u}_n\to u_*$ in $L^2(0,T;H_{\mathrm{loc}})$, $\bar{\mathbb{P}}$-a.s., by the second part of assumption (G.3) we infer that for all $t\in[0,T]$ and all $v\in\mathscr{V}$

$$
\lim_{n\to\infty}\int_0^t \|\langle G(s,\bar{u}_n(s))-G(s,u_*(s))|v\rangle\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\,ds=0.
\tag{81}
$$

Moreover, by Eqs. 36 and 59 we see that for every $t \in [0, T]$ every $r \in \left(1, 2 + \frac{\gamma}{2}\right]$ and every $n \in \mathbb{N}$

$$
\begin{aligned}
&\bar{\mathbb{E}}\left[\left|\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds\right|^r\right] \\
&\le c\,\bar{\mathbb{E}}\left[\|v\|_V^{2r} \cdot \int_0^t \left\{\|G(s, \bar{u}_n(s))\|^{2r}_{\mathcal{L}_{\mathrm{HS}}(Y_W;V')} + \|G(s, u_*(s))\|^{2r}_{\mathcal{L}_{\mathrm{HS}}(Y_W;V')}\right\} ds\right] \\
&\le c_1\,\bar{\mathbb{E}}\left[\int_0^T (1 + |\bar{u}_n(s)|_H^{2r} + |u_*(s)|_H^{2r})\, ds\right] \\
&\le \tilde{c}\left\{1 + \bar{\mathbb{E}}\left[\sup_{s\in[0,T]} |\bar{u}_n(s)|_H^{2r} + \sup_{s\in[0,T]} |u_*(s)|_H^{2r})\right]\right\} \le \tilde{c}(1 + 2C_1(2r))
\end{aligned}
\tag{82}
$$

for some positive constants $c, c_1, \tilde{c}$. Thus by Eqs. 81, 82 and the Vitali Theorem

$$
\lim_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds\right] = 0 \quad \text{for all } v \in \mathcal{V}. \tag{83}
$$

Let now $v \in V$ and let $\varepsilon > 0$. Since $\mathcal{V}$ is dense in V, there exists $v_\varepsilon \in \mathcal{V}$ such that $\|v - v_\varepsilon\|_V \le \varepsilon$. We have the following inequalities

$$
\begin{aligned}
&\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds \\
&\le 2\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v - v_\varepsilon\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds \\
&\quad +2\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v_\varepsilon\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds.
\end{aligned}
$$

Moreover, by inequality (36) in assumption (G.3), we obtain the following estimates

$$
\begin{aligned}
&\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v - v_\varepsilon\rangle\|^2_{\mathcal{L}+(Y_w;\mathbb{R})}\, ds \\
&\le \|v - v_\varepsilon\|_V^2 \int_0^t \|G(s, \bar{u}_n(s)) - G(s, u_*(s))\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;V')}\, ds \\
&\le c\left(1 + \sup_{s\in[0,T]} |\bar{u}_n(s)|_H^2 + \sup_{s\in[0,T]} |u_*(s)|_H^2\right)\varepsilon^2
\end{aligned}
$$

for some $c > 0$. Thus by Eq. 59 we obtain the following inequalities

$$
\begin{aligned}
&\bar{\mathbb{E}}\left[\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds\right] \\
&\le 2c\,\{1 + 2C_1(2)\}\,\varepsilon^2 + 2\bar{\mathbb{E}}\left[\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v_\varepsilon\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds\right].
\end{aligned}
$$

Passing to the upper limit as $n \to \infty$ by Eq. 83 we infer that for all $v \in V$

$$
\limsup_{n\to\infty} \bar{\mathbb{E}}\left[\int_0^t \|\langle G(s, \bar{u}_n(s)) - G(s, u_*(s))|v\rangle\|^2_{\mathcal{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds\right] \le C\varepsilon^2,
$$

where $C = 2c\{1 + 2C_1(2)\}$. Since $\varepsilon > 0$ was chosen in an arbitrary way, we infer that

$$\lim_{n\to\infty} \bar{\mathbb{E}} \left[\int_0^t \| \langle G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle \|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})} \, ds \right] = 0 \quad \text{for all } v \in V. \tag{84}$$

For every $v \in V$ and every $s \in [0, T]$ we have

$$\begin{aligned}
&\langle P_n G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle = \langle G(s, \bar{u}_n(s)) | P_n v \rangle - \langle G(s, u_*(s)) | v \rangle \\
&= \langle G(s, \bar{u}_n(s)) | P_n v - v \rangle + \langle G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle \\
&\leq \| G(s, \bar{u}_n(s)) \|_{\mathscr{L}_{\mathrm{HS}}(Y_W, V')} \| P_n v - v \|_V + \langle G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle.
\end{aligned}$$

Thus by inequality (36) in assumption (G.3) and by Eq. 59 we obtain

$$\begin{aligned}
&\bar{\mathbb{E}} \left[\int_0^t \| \langle P_n G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle \|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})} \, ds \right] \\
&\leq 2 \| P_n v - v \|_V^2 \, \bar{\mathbb{E}} \left[\int_0^T \| G(s, \bar{u}_n(s)) \|^2_{\mathscr{L}_{\mathrm{HS}}(Y_W, V')} \, ds \right] \\
&\quad + 2 \bar{\mathbb{E}} \left[\int_0^t \| \langle G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle \|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})} \, ds \right] \\
&\leq 2C \| P_n v - v \|_V^2 \, \bar{\mathbb{E}} \left[\int_0^T \left(1 + |\bar{u}_n(s)|_H^2 \right) ds \right] \\
&\quad + 2 \bar{\mathbb{E}} \left[\int_0^t \| \langle G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle \|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})} \, ds \right] \\
&\leq 2CT(1 + C_1(2)) \| P_n v - v \|_V^2 \\
&\quad + 2 \bar{\mathbb{E}} \left[\int_0^t \| \langle G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle \|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})} \, ds \right].
\end{aligned}$$

Since $U \subset V$ and $\| P_n v - v \|_V \to 0$ for all $v \in U$, see Section 2.3, by Eq. 84 we infer that

$$\lim_{n\to\infty} \bar{\mathbb{E}} \left[\int_0^t \| \langle \mathbb{P}n G(s, \bar{u}_n(s)) - G(s, u_*(s)) | v \rangle \|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})} \, ds \right] = 0 \quad \text{for all } v \in U.$$

Hence by the properties of the Itô integral we infer that for all $t \in [0, T]$ and all $v \in U$

$$\lim_{n\to\infty} \bar{\mathbb{E}} \left[\left| \left\langle \int_0^t \left[P_n G(s, \bar{u}_n(s)) - G(s, u_*(s)) \right] dW_*(s) \Big| v \right\rangle \right|^2 \right] = 0. \tag{85}$$

Moreover, by the Itô isometry, inequality (36) in assumption (G.3), and Eq. 59 we have for all $t \in [0,T]$ and all $n \in \mathbb{N}$

$$\begin{aligned}
&\bar{\mathbb{E}}\left[\left|\left\langle \int_0^t \left[P_n G(s,\bar{u}_n(s)) - G(s,u_*(s))\right] dW_*(s) \Big| v \right\rangle\right|^2\right] \\
&= \bar{\mathbb{E}}\left[\int_0^t \|\langle P_n G(s,\bar{u}_n(s)) - G(s,u_*(s))|v\rangle\|^2_{\mathscr{L}_{\mathrm{HS}}(Y_w;\mathbb{R})}\, ds\right] \\
&\le c\left\{1 + \bar{\mathbb{E}}\left[\sup_{s\in[0,T]} |\bar{u}_n(s)|_H^2 + \sup_{s\in[0,T]} |u_*(s)|_H^2)\right]\right\} \le c(1+2C_1(2)) \qquad (86)
\end{aligned}$$

for some $c > 0$. By Eqs. 85, 86 and the Dominated Convergence Theorem we infer that

$$\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\left\langle \int_0^t \left[P_n G(s,\bar{u}_n(s)) - G(s,u_*(s))\right] dW_*(s) \Big| v \right\rangle\right|^2\right] = 0. \qquad (87)$$

By Eqs. 67, 70, 74, 80 and 87 the proof of Eq. 64 is complete.

Step 2^0 Since u_n is a solution of the Galerkin equation, for all $t \in [0,T]$

$$(u_n(t)|v)_H = \mathscr{K}_n(u_n,\eta_n,W_n,v)(t), \qquad \mathbb{P}\text{-a.s.}$$

In particular,

$$\int_0^T \mathbb{E}\left[\left|(u_n(t)|v)H - \mathscr{K}_n(u_n,\eta_n,W_n,v)(t)\right|^2\right] dt = 0.$$

Since $\mathscr{L}(u_n,\eta_n,W_n) = \mathscr{L}(\bar{u}_n,\bar{\eta}_n,\bar{W}_n)$,

$$\int_0^T \bar{\mathbb{E}}\left[\left|(\bar{u}_n t|v)_H - \mathscr{K}_n(\bar{u}_n,\bar{\eta}_n,\bar{W}_n,v)(t)\right|^2\right] dt = 0.$$

Moreover, by Eqs. 63 and 64

$$\int_0^T \bar{\mathbb{E}}\left[\left|(u_*(t)|v)_H - \mathscr{K}(u_*,\eta_*,W_*,v)(t)\right|^2\right] dt = 0.$$

Hence for l-almost all $t \in [0,T]$ and $\bar{\mathbb{P}}$-almost all $\omega \in \bar{\Omega}$

$$(u_*(t)|v)_H - \mathscr{K}(u_*,\eta_*,W_*,v)(t) = 0,$$

i.e. for l-almost all $t \in [0,T]$ and $\bar{\mathbb{P}}$-almost all $\omega \in \bar{\Omega}$

$$\begin{aligned}
(u_*(t)|v)_H - (u_*(0)|v)_H &+ \int_0^t \langle \mathscr{A} u_*(s)|v\rangle\, ds + \int_0^t \langle B(u_*(s),u_*(s))|v\rangle\, ds \\
&- \int_0^t \langle f(s)|v\rangle\, ds - \int_0^t \int_Y (F(s,u_*(s);y)|v)_H\, \tilde{\eta}_*(ds,dy) \\
&- \left\langle \int_0^t G(s,u_*(s))\, dW_*(s) \Big| v \right\rangle = 0.
\end{aligned}$$

Since u_* is $\mathscr{Z}$-valued random variable, in particular $u_* \in \mathbb{D}([0,T];H_w)$, i.e. u_* is weakly càdlàg. Hence the function on the left-hand side of the above equality is càdlàg with respect to t. Since two càdlàg functions equal for l-almost all $t \in [0, T]$ must be equal for all $t \in [0, T]$, we infer that for all $t \in [0, T]$ and all $v \in U$

$$(u_*(t)|v)_H - (u_*(0)|v)_H + \int_0^t \langle \mathscr{A} u_*(s)|v\rangle\, ds + \int_0^t \langle B(u_*(s), u_*(s))|v\rangle\, ds$$
$$- \int_0^t \langle f(s)|v\rangle\, ds - \int_0^t \int_Y (F(s, u_*(s); y)|v)_H\, \tilde{\eta}_*(ds, dy)$$
$$- \left\langle \int_0^t G(s, u_*(s))\, dW_*(s) | v \right\rangle = 0 \qquad \bar{\mathbb{P}}\text{-a.s.}$$

Since U is dense in V, we infer that the above equality holds for all $v \in V$. Putting $\bar{u} := u_*$, $\bar{\eta} := \eta_*$ and $\bar{W} := W_*$, we infer that the system $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}}, \bar{u}, \bar{\eta}, \bar{W})$ is a martingale solution of the Eq. (30). The proof of Theorem 3 is thus complete. □

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Appendix A

A.1 The Aldous Condition

Here $(\mathbb{S}, \rho)$ is a separable and complete metric space. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} := (\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual hypotheses, see [21], and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg, $\mathbb{F}$-adapted and $\mathbb{S}$-valued processes.

Definition 4 (See [19]) We say that the sequence (X_n) of $\mathbb{S}$-valued random variables satifies condition $[\tilde{\mathbf{T}}]$ iff

$[\tilde{\mathbf{T}}]$ $\quad \forall\, \varepsilon > 0 \quad \forall\, \eta > 0 \quad \exists\, \delta > 0$:

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left\{ w_{[0,T]}(X_n, \delta) > \eta \right\} \le \varepsilon.$$

Let us recall that $w_{[0,T]}$ stands for the modulus defined by Eq. 21.

Remark Let $\mathbb{P}_n$ denote the law of X_n on $\mathbb{D}([0, T], \mathbb{S})$. For fixed $\eta > 0$ and $\delta > 0$ we denote

$$C_{\eta,\delta} := \{u \in \mathbb{D}([0, T], \mathbb{S}) : \quad w_{[0,T]}(u, \delta) \ge \eta\}.$$

Then condition

$$\mathbb{P}\left\{ w_{[0,T]}(X_n, \delta) > \eta \right\} \le \varepsilon$$

is equivalent to

$$\mathbb{P}_n(C_{\eta,\delta}) \le \varepsilon.$$

Lemma 7 *Assume that* (X_n) *satifies condition* $[\tilde{\mathbf{T}}]$. *Let* $\mathbb{P}_n$ *be the law of* X_n *on* $\mathbb{D}([0,T],\mathbb{S})$, $n \in \mathbb{N}$. *Then for every* $\varepsilon > 0$ *there exists a subset* $A_\varepsilon \subset \mathbb{D}([0,T],\mathbb{S})$ *such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n(A_\varepsilon) \geq 1-\varepsilon$$

and

$$\lim_{\delta\to 0} \sup_{u\in A_\varepsilon} w_{[0,T]}(u,\delta) = 0. \tag{88}$$

Proof Fix $\varepsilon > 0$. By $[\tilde{\mathbf{T}}]$, for each $k \in \mathbb{N}$ there exists $\delta_k > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left\{ w_{[0,T]}(X_n,\delta_k) > \frac{1}{k} \right\} \leq \frac{\varepsilon}{2^{k+1}}.$$

Then

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left\{ w_{[0,T]}(X_n,\delta_k) \leq \frac{1}{k} \right\} \geq 1 - \frac{\varepsilon}{2^{k+1}}$$

or equivalently

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n\left\{ u \in \mathbb{D}([0,T],\mathbb{S}) : \;\; w_{[0,T]}(u,\delta_k) \leq \frac{1}{k} \right\} \geq 1 - \frac{\varepsilon}{2^{k+1}}$$

Let $B_k := \left\{ u \in \mathbb{D}([0,T],\mathbb{S}) : \;\; w_{[0,T]}(u,\delta_k) \leq \frac{1}{k} \right\}$ and let $A_\varepsilon := \bigcap_{k=1}^{\infty} B_k$. We assert that for each $n \in \mathbb{N}$

$$\mathbb{P}_n\left(A_\varepsilon\right) \geq 1-\varepsilon.$$

Indeed, we have the following estimate

$$\begin{aligned}
\mathbb{P}_n\left(\mathbb{D}([0,T],\mathbb{S})\setminus A_\varepsilon\right) &\leq \mathbb{P}_n\left(\mathbb{D}([0,T],\mathbb{S})\setminus \bigcap_{k=1}^{\infty} B_k \right) = \mathbb{P}_n\left(\bigcup_{k=1}^{\infty} (\mathbb{D}([0,T],\mathbb{S}) \setminus B_k) \right) \\
&\leq \sum_{k=1}^{\infty} \mathbb{P}_n\left(\mathbb{D}([0,T],\mathbb{S}) \setminus B_k\right) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \varepsilon.
\end{aligned}$$

Thus $\mathbb{P}_n(A_\varepsilon) \geq 1-\varepsilon$.

To prove Eq. 88, let us fix $\tilde{\varepsilon} > 0$. Directly from the definition of A_ε, we infer that $\sup_{u\in A_\varepsilon} w_{[0,T]}(u,\delta_k) \leq \frac{1}{k}$ for each $k \in \mathbb{N}$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} \leq \tilde{\varepsilon}$ and let $\delta_0 := \delta_{k_0}$. Then for every $\delta \leq \delta_0$ we obtain

$$w_{[0,T]}(u,\delta) \leq w_{[0,T]}(u,\delta_{k_0}) \leq \tilde{\varepsilon}$$

which completes the proof of Eq. 88 and the proof of Lemma. □

Now, we recall the Aldous condition which is connected with condition $[\tilde{\mathbf{T}}]$ (see [19, 22] and [2]). This condition allows to investigate the modulus for the sequence of stochastic processes by means of stopped processes.

Definition 5 A sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition **[A]** iff

[A] $\forall\,\varepsilon>0 \quad \forall\,\eta>0 \quad \exists\,\delta>0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n\le T$ one has

$$\sup_{n\in\mathbb{N}}\sup_{0\le\theta\le\delta}\mathbb{P}\left\{\varrho\left(X_n(\tau_n+\theta),X_n(\tau_n)\right)\ge\eta\right\}\le\varepsilon.$$

Lemma 8 (See [19], Theorem 2.2.2) *Condition* [**A**] *implies condition* [$\tilde{\mathbf{T}}$].

In the following Remark we formulate a certain condition which guaranties that the sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition **[A]**.

Lemma 9 *Let $(E,\|\cdot\|_E)$ be a separable Banach space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of E-valued random variables. Assume that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n\le T$ and for every $n\in\mathbb{N}$ and $\theta\ge0$ the following condition holds*

$$\mathbb{E}\left[(\|X_n(\tau_n+\theta)-X_n(\tau_n)\|_E^{\alpha}\right]\le C\theta^{\beta} \tag{89}$$

for some $\alpha,\beta>0$ and some constant $C>0$. Then the sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition **[A]** in the space E.

Proof Let us fix $\varepsilon>0$ and $\eta>0$. By the Chebyshev inequality for every $n\in\mathbb{N}$ and every $\theta>0$ we have

$$\mathbb{P}\left\{\|X_n(\tau_n+\theta)-X_n(\tau_n)\|_E\ge\eta\right\}\le\frac{1}{\eta^{\alpha}}\mathbb{E}\left[(\|X_n(\tau_n+\theta)-X_n(\tau_n)\|_E^{\alpha}\right]\le\frac{C\theta^{\beta}}{\eta^{\alpha}}.$$

Let $\delta:=\left(\frac{\eta^{\alpha}\varepsilon}{C}\right)^{\frac{1}{\beta}}$. Let us fix $n\in\mathbb{N}$. Then for every $\theta\in[0,\delta]$ we have the following inequalities

$$\mathbb{P}\left\{\|X_n(\tau_n+\theta)-X_n(\tau_n)\|_E\ge\eta\right\}\le\frac{C\theta^{\beta}}{\eta^{\alpha}}\le\frac{C}{\eta^{\alpha}}\left[\left(\frac{\eta^{\alpha}\varepsilon}{C}\right)^{\frac{1}{\beta}}\right]^{\beta}=\varepsilon.$$

Hence

$$\sup_{0\le\theta\le\delta}\mathbb{P}\left\{\|X_n(\tau_n+\theta)-X_n(\tau_n)\|_E\ge\eta\right\}\le\varepsilon.$$

Since the above inequality holds for every $n\in\mathbb{N}$, one has

$$\sup_{n\in\mathbb{N}}\sup_{0\le\theta\le\delta}\mathbb{P}\left\{\|X_n(\tau_n+\theta)-X_n(\tau_n)\|_E\ge\eta\right\}\le\varepsilon,$$

i.e. condition **[A]** is satisfied. This completes the proof. □

A.2 Proof of Corollary 1

Let $\varepsilon>0$. By the Chebyshev inequality and by (a), we infer that for any $r>0$

$$\mathbb{P}\left(\sup_{s\in[0,T]}|X_n(s)|_H>r\right)\le\frac{\mathbb{E}\left[\sup_{s\in[0,T]}|X_n(s)|_H\right]}{r}\le\frac{C_1}{r}.$$

Let R_1 be such that $\frac{C_1}{R_1} \le \frac{\varepsilon}{3}$. Then

$$\mathbb{P}\left(\sup_{s\in[0,T]} |X_n(s)|_H > R_1 \right) \le \frac{\varepsilon}{3}$$

Let $B_1 := \left\{ u \in \mathscr{Z}_q : \ \sup_{s\in[0,T]} |u(s)|_H \le R_1 \right\}$.

By the Chebyshev inequality and by (b), we infer that for any $r > 0$

$$\mathbb{P}\left(\|X_n\|_{L^q(0,T;V)} > r \right) \le \frac{\mathbb{E}\left[\|X_n\|^q_{L^q(0,T;V)} \right]}{r^q} \le \frac{C_2}{r^q}.$$

Let R_2 be such that $\frac{C_2}{R_2^q} \le \frac{\varepsilon}{3}$. Then

$$\mathbb{P}\left(\|X_n\|_{L^q(0,T;V)} > R_2 \right) \le \frac{\varepsilon}{3}.$$

Let $B_2 := \left\{ u \in \mathscr{Z}_q : \ \|u\|_{L^q(0,T;V)} \le R_2 \right\}$.

By Lemmas 7 and 8 there exists a subset $A_{\frac{\varepsilon}{3}} \subset \mathbb{D}([0,T], E_1)$ such that $\tilde{\mathbb{P}}_n\left(A_{\frac{\varepsilon}{3}} \right) \ge 1 - \frac{\varepsilon}{3}$ and

$$\lim_{\delta\to 0} \sup_{u\in A_{\frac{\varepsilon}{3}}} w_{[0,T]}(u,\delta) = 0.$$

It is sufficient to define K_ε as the closure of the set $B_1 \cap B_2 \cap A_{\frac{\varepsilon}{3}}$ in $\mathscr{Z}_q$. By Theorem 2, K_ε is compact in $\mathscr{Z}_q$. The proof is thus complete. □

Appendix B: The Skorokhod Embedding Theorems

Let us recall the following Jakubowski's version of the Skorokhod Theorem [18], see also Brzeźniak and Ondreját [10].

Theorem 4 (Theorem 2 in [18]) *Let $(\mathscr{X}, \tau)$ be a topological space such that there exists a sequence (f_m) of continuous functions $f_m : \mathscr{X} \to \mathbb{R}$ that separates points of $\mathscr{X}$. Let (X_n) be a sequence of $\mathscr{X}$ valued random variables. Suppose that for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathscr{X}$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\{X_n \in K_\varepsilon\}) > 1 - \varepsilon.$$

Then there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$, a sequence $(Y_k)_{k\in\mathbb{N}}$ of $\mathscr{X}$ valued random variables and an $\mathscr{X}$ valued random variable Y defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that

$$\mathscr{L}(X_{n_k}) = \mathscr{L}(Y_k), \qquad k = 1, 2, \ldots$$

and forall $\omega \in \Omega$:

$$Y_k(\omega) \xrightarrow{\tau} Y(\omega) \quad \text{as } k \to \infty.$$

We will use the following version of the Skorokhod Theorem due to Brzeźniak and Hausenblas [6].

Theorem 5 (Theorem E.1 in [6]) *Let E_1, E_2 be two separable Banach spaces and let $\pi_i : E_1 \times E_2 \to E_i$, $i = 1, 2$, be the projection onto E_i, i.e.*

$$E_1 \times E_2 \ni \chi = (\chi_1, \chi_2) \to \pi_i(\chi) \in E_i.$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\chi_n : \Omega \to E_1 \times E_2$, $n \in \mathbb{N}$, be a family of random variables such that the sequence $\{\mathscr{L}aw(\chi_n), n \in \mathbb{N}\}$ is weakly convergent on $E_1 \times E_2$. Finally let us assume that there exists a random variable $\rho : \Omega \to E_1$ such that $\mathscr{L}aw(\pi_1 \circ \chi_n) = \mathscr{L}aw(\rho)$, $\forall\, n \in \mathbb{N}$.

Then there exists a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$, a family of $E_1 \times E_2$-valued random variables $\{\bar{\chi}_n, n \in \mathbb{N}\}$ on $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ and a random variable $\chi_* : \bar{\Omega} \to E_1 \times E_2$ such that

- (i) $\mathscr{L}aw(\bar{\chi}_n) = \mathscr{L}aw(\chi_n)$, $\forall\, n \in \mathbb{N}$;
- (ii) $\bar{\chi}_n \to \chi_*$ *in* $E_1 \times E_2$ *a.s.;*
- (iii) $\pi_1 \circ \bar{\chi}_n(\bar{\omega}) = \pi_1 \circ \chi_*(\bar{\omega})$ *for all* $\bar{\omega} \in \bar{\Omega}$.

Remark Theorem 5 remains true if we substitute the Banach spaces E_1, E_2 by the separable complete metric spaces.

Using the ideas due to Jakubowski [18], we can proof the following generalization of Theorem 5 to the case of nonmetric spaces. Let us notice that in comparison to Theorem 5 we will assume that the sequence $\{\mathscr{L}aw(\chi_n), n \in \mathbb{N}\}$ is tight. The assumption of the weak convergence of $\{\mathscr{L}aw(\chi_n), n \in \mathbb{N}\}$ is not sufficient in the case of nonmetric spaces, see [18].

Corollary 2 (Corollary 5.3 in [25]) *Let $\mathscr{X}_1$ be a separable complete metric space and let $\mathscr{X}_2$ be a topological space such that there exists a sequence $\{f_\iota\}_{\iota \in \mathbb{N}}$ of continuous functions $f_\iota : \mathscr{X}_2 \to \mathbb{R}$ separating points of $\mathscr{X}_2$. Let $\mathscr{X} := \mathscr{X}_1 \times \mathscr{X}_2$ with the Tykhonoff topology induced by the projections*

$$\pi_i : \mathscr{X}_1 \times \mathscr{X}_2 \to \mathscr{X}_i, \qquad i = 1, 2.$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\chi_n : \Omega \to \mathscr{X}_1 \times \mathscr{X}_2$, $n \in \mathbb{N}$, be a family of random variables such that the sequence $\{\mathscr{L}aw(\chi_n), n \in \mathbb{N}\}$ is tight on $\mathscr{X}_1 \times \mathscr{X}_2$. Finally let us assume that there exists a random variable $\rho : \Omega \to \mathscr{X}_1$ such that $\mathscr{L}aw(\pi_1 \circ \chi_n) = \mathscr{L}aw(\rho)$ for all $n \in \mathbb{N}$.

Then there exists a subsequence $\left(\chi_{n_k}\right)_{k \in \mathbb{N}}$, a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$, a family of $\mathscr{X}_1 \times \mathscr{X}_2$-valued random variables $\{\bar{\chi}_k, k \in \mathbb{N}\}$ on $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ and a random variable $\chi_* : \bar{\Omega} \to \mathscr{X}_1 \times \mathscr{X}_2$ such that

- (i) $\mathscr{L}aw(\bar{\chi}_k) = \mathscr{L}aw(\chi_{n_k})$ *for all* $k \in \mathbb{N}$;
- (ii) $\bar{\chi}_k \to \chi_*$ *in* $\mathscr{X}_1 \times \mathscr{X}_2$ *a.s. as* $k \to \infty$;
- (iii) $\pi_1 \circ \bar{\chi}_k(\bar{\omega}) = \pi_1 \circ \chi_*(\bar{\omega})$ *for all* $\bar{\omega} \in \bar{\Omega}$.

For the convenience of the reader we recall the proof.

Proof Using the ideas due to Jakubowski [18], the proof can be reduced to Theorem 5. Let us denote

$$\chi_n = \left(\chi_n^1, \chi_n^2\right),$$

where $\chi_n^i := \pi_i \circ \chi_n$, $i = 1, 2$. Since the sequence $\{\mathscr{L}aw(\chi_n), n \in \mathbb{N}\}$ is tight on $\mathscr{X}_1 \times \mathscr{X}_2$, we infer that the sequence $\{\mathscr{L}aw(\chi_n^2), n \in \mathbb{N}\}$ is tight on $\mathscr{X}_2$. Let $K_m \subset \mathscr{X}_2$ be compact subsets such that $K_m \subset K_{m+1}$, $m = 1, 2, \dots$ and

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\{\chi_n^2 \in K_m\}) > 1 - \frac{1}{m}. \tag{90}$$

Let us consider the mapping $\tilde{f} : \mathscr{X}_2 \to \mathbb{R}^{\mathbb{N}}$ defined by

$$\tilde{f}(z) := (f_1(z), f_2(z), \dots) = (f_\iota(z))_{\iota\in\mathbb{N}}, \qquad z \in \mathscr{X}_2.$$

$$\tilde{\mu}_n := \mathscr{L}(\tilde{f}(\chi_n^2)) \qquad \text{and} \qquad \tilde{K}_m := \tilde{f}(K_m).$$

On the set $\mathbb{R}^{\mathbb{N}}$ let us define the function

$$\Phi(y) := \min\{m : \ y \in \tilde{K}_m\} \quad \text{if } y \in \bigcup_{m=1}^{\infty} \tilde{K}_m$$

$$\Phi(y) = +\infty \qquad \text{otherwise.}$$

Function $\Phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{N}$ is lower semicontinuous, i.e.: if $y_n \to y_0$ in $\mathbb{R}^{\mathbb{N}}$, then

$$\liminf_{n\to\infty} \Phi(y_n) \geq \Phi(y_0)$$

From Eq. 90 it follows that

- $\Phi < \infty$ ($\tilde{\mu}_n$-p.p.) for all $n \in \mathbb{N}$
- and $(\tilde{\mu}_n \circ \Phi^{-1})$ is a tight sequence of laws on $\mathbb{N}$.

Furthermore, the sequence of laws $\left\{\mathscr{L}aw(\tilde{f} \circ \chi_n^2, \Phi \circ \tilde{f} \circ \chi_n^2), n \in \mathbb{N}\right\}$ is tight on $\mathbb{R}^{\mathbb{N}} \times \mathbb{N}$.

Let us consider the product space $\mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N})$ and let $P_1 := \mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N}) \to \mathscr{X}_1$ be the projection onto $\mathscr{X}_1$ and $P_2 := \mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N}) \to \mathbb{R}^{\mathbb{N}} \times \mathbb{N}$ be the projection onto $\mathbb{R}^{\mathbb{N}} \times \mathbb{N}$. Moreover let ξ_n, $n \in \mathbb{N}$, be $\mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N})$-valued random variables defined by

$$\xi_n := \left(\xi_n^1, \xi_n^2\right) : \Omega \to \mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N}),$$

where

$$\xi_n^1 := \chi_n^1 \qquad \text{and} \qquad \xi_n^2 := (\tilde{f} \circ \chi_n^2, \Phi \circ \tilde{f} \circ \chi_n^2), \qquad n \in \mathbb{N}.$$

Remark that the sequence of laws $\left\{\mathscr{L}aw(\xi_n), n \in \mathbb{N}\right\}$ is tight on $\mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N})$. By the Prokhorov Theorem we can choose a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\left\{\mathscr{L}aw(\xi_{n_k}), k \in \mathbb{N}\right\}$ is weakly convergent on $\mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N})$. Thus the subsequence $\left(\xi_{n_k}\right)_{k\in\mathbb{N}}$ satisfies the assumption of Theorem 5. Hence there exists a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$, a family of $\mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N})$-valued random variables $\{\bar{\xi}_k, \ k \in \mathbb{N}\}$ on $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ and a random variable $\xi_* : \bar{\Omega} \to \mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N})$ such that

(i) $\mathscr{L}aw(\bar{\xi}_k) = \mathscr{L}aw(\xi_{n_k})$ for all $k \in \mathbb{N}$;
(ii) $\bar{\xi}_k \to \xi_*$ in $\mathscr{X}_1 \times (\mathbb{R}^{\mathbb{N}} \times \mathbb{N})$ a.s. as $k \to \infty$;
(iii) $P_1 \circ \bar{\xi}_k(\bar{\omega}) = P_1 \circ \xi_*(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$.

Let us put

$$\bar{\chi}_k^1 := P_1 \circ \bar{\xi}_k, \qquad k \in \mathbb{N}.$$

Notice that $\left(P_2 \circ \bar{\xi}_k\right)_{k\in\mathbb{N}}$ is the Skorokhod representation for the sequence $\left(\tilde{f} \circ \chi_{n_k}^2, \Phi \circ \tilde{f} \circ \chi_{n_k}^2\right)_{k\in\mathbb{N}}$. Let $P_2 \circ \bar{\xi}_k = (\eta_k^1, \eta_k^2)$, where $\eta_k^1 : \bar{\Omega} \to \mathbb{R}^{\mathbb{N}}$ and $\eta_k^2 : \bar{\Omega} \to \mathbb{N}$, $k \in \mathbb{N}$ and let $P_2 \circ \xi_* = (\eta_*^1, \eta_*^2)$, where $\eta_*^1 : \bar{\Omega} \to \mathbb{R}^{\mathbb{N}}$ and $\eta_*^2 : \bar{\Omega} \to \mathbb{N}$. In the same way as in the proof of Lemma 1 in [18], we can prove that $\eta_k^2 = \Phi(\eta_k^1)$, $\bar{\mathbb{P}}$-a.s., $k \in \mathbb{N}$. Since $\eta_*^2 < \infty$ $\bar{\mathbb{P}}$-a.s., we have

$$\sup_{k\in\mathbb{N}} \Phi(\eta_k^1) < \infty \qquad \bar{\mathbb{P}}\text{-a.s.}$$

Thus for $\bar{\mathbb{P}}$-almost all $\omega \in \bar{\Omega}$ the values $\eta_k^1(\omega)$ belong to the σ-compact subspace $\bigcup_{m=1}^{\infty} \tilde{K}_m = \tilde{f}\left(\bigcup_{m=1}^{\infty} K_m\right)$. Since $\tilde{f}$ restricted to σ-compact subspace is a measurable homeomorphism, we can define

$$\bar{\chi}_k^2 := \tilde{f}^{-1}(\eta_k^1), \qquad k \in \mathbb{N}.$$

Finally $\bar{\chi}_k$ is defined by

$$\bar{\chi}_k := \left(\bar{\chi}_k^1, \bar{\chi}_k^2\right), \qquad k \in \mathbb{N}.$$

This completes the proof. □

In Section 5 we use Corollary 2 for the space

$$\mathscr{X}_2 := \mathscr{Z} := L^2_w(0,T;V) \cap L^2(0,T;H_{\mathrm{loc}}) \cap \mathbb{D}([0,T];U') \cap \mathbb{D}([0,T];H_w).$$

So, in the following Remark we will discuss the problem of existence of the countable family of real valued continuous mappings defined on $\mathscr{Z}$ and separating points of this space.

Remark 2

(1) Since $L^2(0,T;H_{\mathrm{loc}})$ and $\mathbb{D}([0,T];U')$ are separable and completely metrizable spaces, we infer that on each of these spaces there exists a countable family of continuous real valued mappings separating points, see [3], exposé 8.

(2) For the space $L^2_w(0,T;V)$ it is sufficient to put

$$f_m(u) := \int_0^T ((u(t)v_n(t)\, dt \in \mathbb{R}, \qquad u \in L^2(0,T;V), \quad m \in \mathbb{N},$$

where $\{v_m, m \in \mathbb{N}\}$ is a dense subset of $L^2(0,T;V)$. Then $(f_m)_{m\in\mathbb{N}}$ is a sequence of continuous real valued mappings separating points of the space $L^2_w(0,T;V)$.

(3) Let $H_0 \subset H$ be a countable and dense subset of H. Then by Eq. 25 for each $h \in H_0$ the mapping

$$\mathbb{D}([0,T];H_w) \ni u \mapsto (u(\cdot)|h)_H \in \mathbb{D}([0,T];\mathbb{R})$$

is continuous. Since $\mathbb{D}([0,T];\mathbb{R})$ is a separable complete metric space, there exists a sequence $(g_l)_{l\in\mathbb{N}}$ of real valued continuous functions defined on

$\mathbb{D}([0,T];\mathbb{R})$ separating points of this space. Then the mappings $f_{h,l}$, $h \in H_0$, $l \in \mathbb{N}$ defined by

$$f_{h,l}(u) := g_l\left((u(\cdot)|h)_H\right), \qquad u \in \mathbb{D}([0,T];H_w),$$

form a countable family of continuous mappings on $\mathbb{D}([0,T];H_w)$ separating points of this space.

Appendix C: Some Auxilliary Results from Functional Analysis

The following result can be found in Holly and Wiciak, [16]. We recall it together with the proof.

Lemma 10 (see Lemma 2.5, p.99 in [16]) *Consider a separable Banach space Φ having the following property*

$$\textit{there exists a Hilbert space } \mathbb{H} \textit{ such that } \Phi \subset \mathbb{H} \textit{ continuously.} \tag{91}$$

Then there exists a Hilbert space $\left(\mathscr{H},(\cdot|\cdot)_{\mathscr{H}}\right)$ such that $\mathscr{H} \subset \phi$, $\mathscr{H}$ is dense in Φ and the embedding $\mathscr{H} \hookrightarrow \Phi$ is compact.

Proof Without loss of generality we can assume that $\dim \Phi = \infty$ and Φ is dense in $\mathbb{H}$. Since Φ is separable, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \Phi$ linearly dense in Φ. Since Φ is dense in $\mathbb{H}$ and the embedding $\Phi \hookrightarrow \mathbb{H}$ is continuous, the subspace $span\{\varphi_1,\varphi_2,...\}$ is dense in $\mathbb{H}$. After the orthonormalisation of (φ_n) in the Hilbert space $(\mathbb{H},(\cdot|\cdot)_{\mathbb{H}})$ we obtain an orthonormal basis (h_n) of this space. Furthermore, the sequence (h_n) is linearly dense in Φ. Since the natural embedding $\iota : \Phi \hookrightarrow \mathbb{H}$ is continuous, we infer that

$$1 = |h_n|_{\mathbb{H}} = |\iota(h_n)|_{\mathbb{H}} \le |\iota| \cdot |h_n|_\Phi$$

and

$$\frac{1}{|h_n|_\Phi} \le |\iota| \qquad \text{for all } n \in \mathbb{N}.$$

Let us take $\eta_0 \in (0,1)$ and define inductively a sequence $(\eta_n)_{n\in\mathbb{N}}$ by

$$\eta_n := \frac{\eta_{n-1}+1}{2}, \qquad n = 1,2,...$$

The sequence (η_n) is strongly increasing and $\lim_{\to\infty}\eta_n = 1$. Let us define a sequence $(r_n)_{n\in\mathbb{N}}$ by

$$r_n := \frac{1-\eta_n}{2|h_n|_\Phi} > 0, \qquad n = 1,2,...$$

Obviously $\lim_{\to\infty} r_n = 0$. Let us consider the set

$$\mathscr{H} := \left\{x \in \mathbb{H} : \quad \sum_{n=1}^{\infty} \frac{1}{r_n^2} \cdot |(x|h_n)_{\mathbb{H}}|^2 < \infty\right\}$$

and the Hilbert space $L^2_\mu(\mathbb{N}^*, \mathbb{R})$, where $\mu : 2^{\mathbb{N}^*} \to [0, \infty]$ is the measure given by the formula

$$\mu(M) := \sum_{n \in M} \frac{1}{r_n^2}, \qquad M \subset \mathbb{N}^*.$$

The linear operator

$$l : L^2_\mu(\mathbb{N}^*, \mathbb{R}) \ni \xi \mapsto \sum_{n=1}^{\infty} \xi_n h_n \in \mathbb{H}$$

is well defined. Moreover, l is an injection and hence we may introduce the following inner product

$$(\cdot|\cdot)_{\mathscr{H}} := (\cdot|\cdot)_{L^2} \circ l^{-1} : \mathscr{H} \times \mathscr{H} \ni (x, y) \mapsto (l^{-1}x|l^{-1}y)_{L^2} \in \mathbb{R}.$$

Now, l is an isometry onto the pre-Hilbert space $(\mathscr{H}, (\cdot|\cdot)_{\mathscr{H}})$ and consequently $\mathscr{H}$ is $(\cdot|\cdot)_{\mathscr{H}}$-complete. Let us notice that for all $x, y \in \mathscr{H}$

$$(x|y)_{\mathscr{H}} = \sum_{n=1}^{\infty} \frac{1}{r_n^2} \cdot (x|h_n)_{\mathbb{H}}(y|h_n)_{\mathbb{H}}, \qquad |x|^2_{\mathscr{H}} = \sum_{n=1}^{\infty} \frac{1}{r_n^2} \cdot |(x|h_n)_{\mathbb{H}}|^2$$

We will show that $\mathscr{H} \subset \Phi$ continuously. Indeed, let $x \in \mathscr{H}$, $|x|_{\mathscr{H}} \le 1$. Then for each $i \in \mathbb{N}$

$$|(x|h_i)h_i|_\Phi = |(x|h_i)| \cdot |h_i|_\Phi \le r_i|h_i|_\Phi = \frac{1-\eta_{i-1}}{2|h_i|_\Phi}|h_i|_\Phi = \frac{1-\eta_{i-1}}{2} = \eta_i - \eta_{i-1}.$$

Thus, for any $k, n \in \mathbb{N}$, $k < n$, we have the following estimate

$$\left|\sum_{i=k+1}^{n} (x|h_i)h_i\right|_\Phi \le \sum_{i=k+1}^{n} (\eta_i - \eta_{i-1}) = \eta_n - \eta_k.$$

Since in particular, the sequence $\left(s_n := \sum_{i=1}^n (x|h_i)h_i\right)$ is Cauchy in the Banach space $(\Phi, |\cdot|_\Phi)$, there exists $\varphi \in \Phi$ such that $lim_n|s_n - \varphi|_\Phi = 0$. On the other hand, $s_n = \sum_{i=1}^n (x|h_i)h_i \to x$ in $\mathbb{H}$. Thus by the uniqueness of the limit $\varphi = x \in \Phi$ and

$$\sum_{i=1}^{n} (x|h_i)h_i \to x \quad \text{in } \Phi\,.$$

Moreover,

$$|x|_\Phi \overset{\infty \leftarrow n}{\longleftarrow} |s_n|_\Phi \le \eta_n - \eta_0 \overset{n\to\infty}{\longrightarrow} 1 - \eta_0.$$

Thus $\mathscr{H} \subset \Phi$ continuously (with the norm of the embedding not exceeding $1 - \eta_0$). We will show that the embedding $j : \mathscr{H} \hookrightarrow \Phi$ is compact. It is sufficient to prove that the ball $Z := \{x \in \mathscr{H} : |x|_{\mathscr{H}} \le 1\}$ is relatively compact in $(\Phi, |\cdot|_\Phi)$. According to the Hausdorff Theorem it is sufficient to find (for every fixed ε) an ε-net of the set $j(Z)$.

Since $\lim_{n\to\infty} \eta_n = 1$, there exists $n \in \mathbb{N}$ such that $1 - \eta_n \le \frac{\varepsilon}{2}$. The linear operator

$$S_n : \mathscr{H} \ni x \mapsto \sum_{i=1}^{n} (x|h_i)h_i \in \Phi$$

being finite-dimensional is compact. Therefore $S_n(Z)$ is relatively compact in $(\Phi, |\cdot|_\Phi)$ and consequently there is a finite subset $F \subset \Phi$ such that $S_n(Z) \subset \bigcup_{\varphi \in Z} \mathbb{B}_\Phi(\varphi, \frac{\varepsilon}{2})$.

We will show that the set F is the ε-net for $j(Z)$. Indeed, let $x \in Z$. Then $S_N(x) \to x$ in $(\Phi, |\cdot|_\Phi)$ and

$$|x - S_n(x)|_\Phi \overset{\infty \leftarrow N}{\longleftarrow} |S_N(x) - S_n(x)|_\Phi \leq \eta_N - \eta_n \overset{N \to \infty}{\longrightarrow} 1 - \eta_n \leq \frac{\varepsilon}{2}.$$

On the other hand, $S_n(x) \in S_n(Z)$, so, there is $\varphi \in F$ such that $S_n(x) \in \mathbb{B}_\Phi(\varphi, \frac{\varepsilon}{2})$. Finally,

$$|x - \varphi|_\Phi \leq |x - S_n(x)|_\Phi + |S_n(x) - \varphi|_\Phi \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e. $x \in \mathbb{B}_\Phi(\varphi, \varepsilon)$. Thus

$$Z \subset \bigcup_{\varphi \in \Phi} \mathbb{B}_\Phi(\varphi, \varepsilon).$$

The proof is thus complete. □

Appendix D: Proof of Lemma 6

Proof Assume first that $\psi \in \mathscr{V}$. Then there exists $R > 0$ such that $supp\psi$ is a compact subset of $\mathscr{O}_R$. Then, using the integration by parts formula, we infer that for every $v, w \in H$

$$\begin{aligned} |\langle B(v, w)|\psi| &= \left| \int_{\mathscr{O}_R} (v \cdot \nabla \psi) w \, dx \right| \\ &\leq \|u\|_{L^2(\mathscr{O}_R)} \|w\|_{L^2(\mathscr{O}_R)} \|\nabla \psi\|_{L^\infty(\mathscr{O}_R)} \leq C|u|_{H_{\mathscr{O}_R}} |w|_{H_{\mathscr{O}_R}} \|\psi\|_{V_m}. \end{aligned} \tag{92}$$

We have $B(u_n, u_n) - B(u, u) = B(u_n - u, u_n) + B(u, u_n - u)$. Thus, using the estimate (92) and the Hőlder inequality, we obtain

$$\begin{aligned} &\left| \int_0^t \langle B\left(u_n(s), u_n(s)\right)|\psi\rangle \, ds - \int_0^t \langle B\left(u(s), u(s)\right)|\psi\rangle \, ds \right| \\ &\leq \left| \int_0^t \langle B\left(u_n(s) - u(s), u_n(s)\right)|\psi\rangle \, ds \right| + \left| \int_0^t \langle B\left(u(s), u_n(s) - u(s)\right)|\psi\rangle \, ds \right| \\ &\leq C \cdot \|u_n - u\|_{L^2(0,T;H_{\mathscr{O}_R})} \left(\|u_n\|_{L^2(0,T;H_{\mathscr{O}_R})} + \|u\|_{L^2(0,T;H_{\mathscr{O}_R})} \right) \|\psi\|_{V_m}, \end{aligned}$$

where C stands for some positive constant. Since $u_n \to u$ in $L^2(0, T; H_{\mathrm{loc}})$, we infer that for all $\psi \in \mathscr{V}$

$$\lim_{n \to \infty} \int_0^t \langle B\left(u_n(s)\right)|\psi\rangle \, ds = \int_0^t \langle B\left(u(s)\right)|\psi\rangle \, ds. \tag{93}$$

If $\psi \in V_m$ then for every $\varepsilon > 0$ there exists $\psi_\varepsilon \in \mathscr{V}$ such that $\|\psi - \psi_\varepsilon\|_{V_m} \le \varepsilon$. Then

$$\begin{aligned}
&|\langle B(u_n(s)) - B(u(s))|\psi\rangle| = |\langle B(u_n(s)) - B(u(s))|\psi - \psi_\varepsilon\rangle| \\
&\quad + |\langle B(u_n(s)) - B(u(s))|\psi_\varepsilon\rangle| \\
&\le \left(|B(u_n(s))|_{V'_m} + |B(u(s))|_{V'_m}\right) \cdot \|\psi - \psi_\varepsilon\|_{V_m} \\
&\quad + |\langle B(u_n(s)) - B(u(s))|\psi_\varepsilon\rangle| \\
&\le \varepsilon \left(|u_n(s)|_H^2 + |u(s)|_H^2\right) + |\langle B(u_n(s)) - B(u(s))|\psi_\varepsilon\rangle| .
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\int_0^t \langle B(u_n(s)) - B(u(s))|\psi\rangle \, ds\right| \\
&\le \varepsilon \int_0^t \left(|u_n(s)|_H^2 + |u(s)|_H^2\right) ds + \left|\int_0^t \langle B(u_n(s)) - B(u(s))|\psi_\varepsilon\rangle ds\right| \\
&\le \varepsilon \cdot \left(\sup_{n\ge 1} \|u_n\|^2_{L^2(0,T;H)} + \|u\|^2_{L^2(0,T;H)}\right) + \left|\int_0^t \langle B(u_n(s)) - B(u(s))|\psi_\varepsilon\rangle ds\right| .
\end{aligned}$$

Passing to the upper limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \left|\int_0^t \langle B(u_n(s)) - B(u(s))|\psi\rangle \, ds\right| \le M\varepsilon,$$

where $M := \sup_{n\ge 1} \|u_n\|^2_{L^2(0,T;H)} + \|u\|^2_{L^2(0,T;H)} < \infty$. Since $\varepsilon > 0$ is arbitrary, we infer that Eq. 93 holds for all $\psi \in V_m$. The proof of the Lemma 6 is thus complete. □

Appendix E: Proof of Lemma 2

To prove that $u_n \in \mathbb{D}([0,T]; H_w)$, it is sufficient to show that for every $h \in H$ the real-valued functions $(u_n(\cdot)|h)$ are càdlàg on $[0,T]$, i.e. are right continuous and have left limits at every $t \in [0,T]$. Let us fix $n \in \mathbb{N}$ and $t_0 \in [0,T]$ and let us assume that $h \in U$. Since $u_n \in \mathbb{D}([0,T]; U')$, there exists $a \in U'$ such that

$$\lim_{t\to t_0^-} \|u_n(t) - a\|_{U'} = 0. \tag{94}$$

In fact, $a \in H$. Indeed, by assumption (i) $u_n([0,T]) \subset H$ and

$$\sup_{s\in[0,T]} |u_n(s)|_H \le r.$$

Let $(t_k)_{k\in\mathbb{N}} \subset [0,T]$ be a sequence convergent to t_0^-. Since $|u_n(t_k)|_H \le r$, by the Banach–Alaoglu Theorem there exists a subsequence convergent weakly in H to some $b \in H$, i.e. there exists $(t_{k_l})_{l\in\mathbb{N}}$ such that $u_n(t_{k_l}) \to b$ weakly in H as $l \to \infty$. Since the embedding $H \hookrightarrow U'$ is continuous, we infer that

$$u_n(t_{k_l}) \to b \qquad \text{weakly in } U' \text{ as } l \to \infty.$$

On the other hand, by Eq. 94

$$u_n(t_{k_l}) \to a \qquad \text{in } U' \text{ as } l \to \infty.$$

Hence $a = b \in H$.

We have

$$|(u_n(t) - a|h)_H| = |\langle u_n(t) - a|h\rangle| \leq \|u_n(t) - a\|_{U'} \cdot \|h\|_U. \tag{95}$$

By Eqs. 94 and 95 we infer that $\lim_{t\to t_0}(u_n(t) - a|h)_H = 0$. Now, let $h \in H$ and let $\varepsilon > 0$. Since U is dense in H, there exists $h_\varepsilon \in U$ such that $|h - h_\varepsilon|_H \leq \varepsilon$. We have the following inequalities

$$\begin{aligned}|(u_n(t) - a|h)_H| &\leq |(u_n(t) - a|h - h_\varepsilon)_H| + |(u_n(t) - a|h_\varepsilon)_H| \\ &\leq |u_n(t) - a|_H\,|h - h_\varepsilon|_H + |(u_n(t) - a|h_\varepsilon)_H| \\ &\leq 2\varepsilon\|u_n\|_{L^\infty(0,T;H)} + |(u_n(t) - a|h_\varepsilon)_H| \\ &\leq 2\varepsilon r + |(u_n(t) - a|h_\varepsilon)_H|\,.\end{aligned}$$

Passing to the upper limit as $t \to t_0^-$, we obtain

$$\limsup_{t\to t_0^-} |(u_n(t) - a|h)_H| \leq 2\varepsilon r$$

Since ε was chosen in an arbitrary way, we infer that

$$\lim_{t\to t_0^-}(u_n(t) - a|h)_H = 0.$$

The proof of right continuity of u_n is analogous.

We claim that

$$u_n \to u \quad \text{in} \quad \mathbb{D}([0,T];\mathbb{B}_w) \quad \text{as} \quad n \to \infty,$$

i.e. that for all $h \in H$

$$(u_n|h)_H \to (u|h)_H \quad \text{in} \quad \mathbb{D}([0,T];\mathbb{R}).$$

By (ii) and Remark 1 there exists a sequence $(\lambda_n) \subset \Lambda_T$ converging to identity uniformly on $[0,T]$ and such that

$$u_n \circ \lambda_n \to u \quad \text{in} \quad U'$$

uniformly on $[0,T]$. We will prove that for all $h \in H$

$$(u_n \circ \lambda_n|h)_H \to (u|h)_H \quad \text{in} \quad \mathbb{R} \tag{96}$$

uniformly on $[0,T]$.

Indeed, let us first fix $h \in U$. Then for all $s \in [0,T]$ we have

$$|(u_n\circ\lambda_n(s) - u(s)|h)_H| = |\langle u_n\circ\lambda_n(s) - u(s)|h\rangle| \leq \|u_n\circ\lambda_n(s) - u(s)\|_{U'}\|h\|_U.$$

By Remark 1

$$\sup_{s\in[0,T]} |\langle u_n \circ \lambda_n(s) - u(s)|h\rangle| \leq \sup_{s\in[0,T]} \|u_n \circ \lambda_n(s) - u(s)\|_{U'} \cdot \|h\|_U \to 0$$

as $n \to \infty$. Moreover, since U is dense in H, the desired convergence holds for all $h \in H$. Indeed, let us fix $h \in H$ and $\varepsilon > 0$. There exists $h_\varepsilon \in U$ such that $|h - h_\varepsilon|_H \leq \varepsilon$. Using (i), we infer that for all $s \in [0, T]$ the following estimates hold

$$\begin{aligned}
&|(u_n \circ \lambda_n(s) - u(s)|h)_H| \\
&\leq |u_n \circ \lambda_n(s) - u(s)|_H |h - h_\varepsilon|_H + |(u_n \circ \lambda_n(s) - u(s)|h_\varepsilon)_H| \\
&\leq \varepsilon \cdot \|u_n \circ \lambda_n - u\|_{L^\infty(0,T;H)} + |(u_n \circ \lambda_n(s) - u(s)|h_\varepsilon)_H| \\
&\leq 2\varepsilon \cdot \sup_{n \in \mathbb{N}} \|u_n\|_{L^\infty(0,T;H)} + |(u_n \circ \lambda_n(s) - u(s)|h_\varepsilon)_H| \\
&\leq 2\varepsilon r + \sup_{s \in [0,T]} |(u_n \circ \lambda_n(s) - u(s)|h_\varepsilon)_H| .
\end{aligned}$$

Thus

$$\sup_{s \in [0,T]} |(u_n \circ \lambda_n(s) - u(s)|h)_H| \leq 2\varepsilon r + \sup_{s \in [0,T]} |(u_n \circ \lambda_n(s) - u(s)|h_\varepsilon)_H| .$$

Passing to the upper limit as $n \to \infty$ we obtain

$$\limsup_{n \to \infty} \sup_{s \in [0,T]} |(u_n \circ \lambda_n(s) - u(s)|h)_H| \leq 2r\varepsilon.$$

Since ε was chosen in an arbitrary way, we get

$$\lim_{n \to \infty} \sup_{s \in [0,T]} |(u_n \circ \lambda_n(s) - u(s)|h)| = 0.$$

Since $\mathbb{D}([0, T]; \mathbb{B}_w)$ is a complete metric space, we infer that $u \in \mathbb{D}([0, T]; \mathbb{B}_w)$ as well. By Remark 1 this completes the proof of Eq. 96 and of Lemma 2. □

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